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Goodness-of-fit tests for randomly censored Weibull distributions with estimated parameters

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Abstract

We consider goodness-of-fit test statistics for Weibull distributions when data are randomly censored and the parameters are unknown. Koziol and Green (*Biometrika*, 63, 465–474, 1976) proposed the Cramér-von Mises statistic's randomly censored version for a simple hypothesis based on the Kaplan-Meier product limit of the distribution function. We apply their idea to the other statistics based on the empirical distribution function such as the Kolmogorov-Smirnov and Liao and Shimokawa (*Journal of Statistical Computation and Simulation*, 64, 23–48, 1999) statistics. The latter is a hybrid of the Kolmogorov-Smirnov, Cramér-von Mises, and Anderson-Darling statistics. These statistics as well as the Koziol-Green statistic are considered as test statistics for randomly censored Weibull distributions with estimated parameters. The null distributions depend on the estimation method since the test statistics are not distribution free when the parameters are estimated. Maximum likelihood estimation and the graphical plotting method with the least squares are considered for parameter estimation. A simulation study enables the Liao-Shimokawa statistic to show a relatively high power in many alternatives; however, the null distribution heavily depends on the parameter estimation. Meanwhile, the Koziol-Green statistic provides moderate power and the null distribution does not significantly change upon the parameter estimation.

Keywords: Cramér-von Mises, goodness-of-fit tests, Kaplan-Meier estimator, Kolmogorov-Smirnov, maximum likelihood estimator, random censoring, Weibull distribution

1. Introduction

In the statistical analysis of life time data, the goodness-of-fit test procedure is important to choose the distribution that adequately fits the data. Classical goodness-of-fit tests are usually based on graphical analysis, moments such as skewness or kurtosis, chi-squared type, the empirical distribution function (EDF), or regression and correlations. Many studies are conducted so that these analyses are generalized to censored data.

Akritas (1988), Hollander and Peña (1992) generalized chi-squared test statistics to censored cases. Koziol and Green (1976), Koziol (1980), and Nair (1981) adapted test statistics based on EDF or weighted empirical process to randomly censored data. The Koziol-Green statistic is a generalized version of the Cramér-von Mises statistic for randomly censored data based on the Kaplan-Meier product limit estimator of the distribution function. Chen (1984) considered a correlation statistic for randomly censored data.

Kim (2012) studied the Koziol-Green and Kolmogorov-Smirnov statistics for a randomly censored exponential distribution with an unknown scale parameter. In this paper, we apply these statistics for

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Weibull distributions when data are randomly censored and the parameters are unknown. Liao and Shimokawa (1999) proposed a new statistic based on the Kolmogorov-Smirnov, Cramér-von Mises and Anderson-Darling statistics and applied it to test Weibull distributions with estimated parameters for a complete sample. Liao and Shimokawa (1999) considered two procedures for the estimation: the first is widely used maximum likelihood estimators (MLEs) and the second is estimators based on graphical plotting method with the least squares on Weibull probability paper. We will generalize their new statistic as well as classical ones to randomly censored Weibull distributions with estimated parameters.

Section 2 presents the test statistics for randomly censored Weibull distributions. Section 3 describes the parameter estimation procedures. Section 4 contains a power comparison of the proposed statistics. Section 4 ends the paper with some conclusion remarks.

2. Goodness-of-fit test statistics

Let $X_1^0, \ldots, X_n^0$ denote lifetimes with distribution function F. The censoring times $C_1, \ldots, C_n$ drawn independently of the $X_1^0, \ldots, X_n^0$ are from distribution function G. The X_i^0's are censored on the right by C_i, so we observe n i.i.d. random pairs (X_i, δ_i), $i = 1, \ldots, n$, where

$$X_i = \min\left(X_i^0, C_i\right) \quad \text{and} \quad \delta_i = \begin{cases} 1, & \text{if } X_i = X_i^0, \\ 0, & \text{if } X_i = C_i. \end{cases} \tag{2.1}$$

We assume that X_i's are the ordered observations without loss of generality for the observed random pairs (X_i, δ_i) in (2.1).

Suppose we wish to test the null hypothesis

$$H_0 : F = F^0 \tag{2.2}$$

with F^0 a specified continuous distribution. When F^0 is fully given in the null hypothesis, we can take the probability integral transformation. Hence we can assume the null hypothesis is $X_1^0, \ldots, X_n^0$ are from uniform distribution on $(0, 1)$, $U(0, 1)$, and F^0 is $U(0, 1)$. We use the product limit estimator $\hat{F}_n^0$ to estimate F^0 since censored data do not have the full knowledge of the empirical distribution.

$$1 - \hat{F}_n^0(t) = \begin{cases} 1, & t < X_1, \\ \prod_{X_j \le t} \left(\dfrac{n-j}{n-j+1}\right)^{\delta_j}, & t < X_n, \\ 0, & t \ge X_n \end{cases} \tag{2.3}$$

The Kaplan-Meier estimator in (2.3) is studied in Kaplan and Meier (1958), Efron (1967), Meier (1975), and Breslow and Crowley (1974). It can also be written as

$$1 - \hat{p}_i = \prod_{j \le i} \left(\frac{n-j}{n-j+1}\right)^{\delta_j}, \quad i = 1, \ldots, n. \tag{2.4}$$

By Michael and Schucany (1986), the Kaplan-Meier estimator in (2.3) could be modified by

$$\hat{F}_{n,c}^0(t) = 1 - \frac{n-c+1}{n-2c+1} \prod_{X_j \le t} \left(\frac{n-j-c+1}{n-j-c+2}\right)^{\delta_j}, \quad 0 \le c \le 1, \tag{2.5}$$

or

$$\hat{p}_{i,c} = 1 - \frac{n-c+1}{n-2c+1} \prod_{j \le i} \left(\frac{n-j-c+1}{n-j-c+2} \right)^{\delta_j}, \quad 0 \le c \le 1,\ i = 1, \dots, n, \tag{2.6}$$

and it reduces to $(i-c)/(n-2c+1)$ for a complete sample. The popular value is $c = 0$ or $c = 0.5$.

Using (2.3) and (2.4), the Kolmogorov-Smirnov statistic based on the EDF could be generalized to randomly censored data by

$$D_n = \sup_t \left| \hat{F}_n^0(t) - t \right| = \max_{1 \le j \le n+1,\, \delta_j = 1} \left\{ \hat{p}_j - X_j, X_j - \hat{p}_{j-1} \right\} \tag{2.7}$$

with $X_{n+1} = 1$, $\hat{p}_0 = 0$, $\hat{p}_{n+1} = 1$. Koziol (1980) proposed a similar statistic to D_n based on the weighted empirical process.

Another popular EDF based test statistic is the Cramér-von Mises statistic. Koziol and Green (1976) generalized it to

$$\psi_n^2 = n \int_0^1 \left(\hat{F}_n^0(t) - t \right)^2 dt \tag{2.8}$$

for randomly censored data using the Kaplan-Meier estimator. It measures the discrepancy between $\hat{F}_n^0$ and $U(0,1)$. By Koziol and Green (1976), the statistic in (2.8) can be rewritten as

$$\psi_n^2 = n \sum_{j=1,\delta_j=1}^{n+1} \hat{F}_n^0\left(X_{j-1}\right)\left(X_j - X_{j-1}\right)\left\{\hat{F}_n^0\left(X_{j-1}\right) - (X_j + X_{j-1})\right\} + \frac{1}{3}n \tag{2.9}$$

with $X_0 = 1$, $X_{n+1} = 1$.

The statistics D_n, ψ_n^2 in (2.7), (2.9) can be trivially modified to $D_{n,c}$, $\psi_{n,c}^2$ using $\hat{F}_{n,c}^0$ or $\hat{p}_{i,c}$ in (2.5), (2.6).

In this paper, we want to test if the lifetimes $Y_1^0, \dots, Y_n^0$ with distribution function F follow a Weibull distribution Weibull(θ, α) for some $\theta > 0$ and $\alpha > 0$. F^0 in the null hypothesis in (2.2) is therefore

$$F^0(y; \theta, \alpha) = 1 - e^{-\left(\frac{y}{\theta}\right)^\alpha}, \quad y > 0, \tag{2.10}$$

and the probability density function (pdf) f^0 is

$$f^0(y; \theta, \alpha) = \frac{\alpha}{\theta^\alpha} y^{\alpha-1} e^{-\left(\frac{y}{\theta}\right)^\alpha}, \quad y > 0. \tag{2.11}$$

Note that $V_i^0 = \ln Y_i^0$ follows the type I extreme value distribution of the minimum $EV(\mu, \sigma)$ with distribution function and pdf,

$$F_E^0(v; \mu, \sigma) = 1 - \exp\left[-\exp\left(\frac{v-\mu}{\sigma} \right) \right], \quad -\infty < v < \infty, \tag{2.12}$$

$$f_E^0(v; \mu, \sigma) = \frac{1}{\sigma} \exp\left[\frac{v-\mu}{\sigma} - \exp\left(\frac{v-\mu}{\sigma} \right) \right], \quad -\infty < v < \infty, \tag{2.13}$$

where μ and σ are location and scale parameters respectively, and

$$\mu = \ln \theta, \qquad \sigma = \frac{1}{\alpha}. \tag{2.14}$$

If we let $T_1, \ldots, T_n$ be the censoring times and let $Y_i = \min(Y_i^0, T_i)$ be the observed random variables, then δ_i can be defined by the same way as in (2.1), and we have the observed random pairs (Y_i, δ_i), $i = 1, \ldots, n$. We can take

$$X_i^0 = F^0\left(Y_i^0\right) = 1 - e^{-\left(Y_i^0/\theta\right)^{\alpha}}$$

and use the test statistic (2.7) or (2.9) for the simple null hypothesis with the parameters θ and α given. However, the null hypothesis is usually

$$H_0 : F(y) = F^0(y; \theta, \alpha), \quad \text{for some } \theta > 0 \text{ and } \alpha > 0,$$

i.e., it is composite and includes some unknown parameters. We therefore need to estimate the unknown parameters θ and α.

If we estimate θ and α by $\hat{\theta}$ and $\hat{\alpha}$, we can take

$$\hat{X}_i^0 = F^0\left(Y_i^0; \hat{\theta}, \hat{\alpha}\right) \equiv \hat{F}^0\left(Y_i^0\right) = 1 - e^{-\left(Y_i^0/\hat{\theta}\right)^{\hat{\alpha}}} \tag{2.15}$$

and consider the similar test statistic $\hat{D}_{n,c}$, $\hat{\psi}_n^2$ as in (2.7) and (2.9). They are as follows.

$$\hat{D}_{n,c} = \max_{1 \le j \le n+1,\, \delta_j = 1} \left\{\hat{p}_{j,c} - \hat{X}_j^0, \hat{X}_j^0 - \hat{p}_{j-1,c}\right\}, \tag{2.16}$$

$$\hat{\psi}_{n,c}^2 = n \sum_{j=1,\delta_j=1}^{n+1} \hat{F}_{n,c}^0\left(\hat{X}_{j-1}^0\right)\left(\hat{X}_j^0 - \hat{X}_{j-1}^0\right)\left\{\hat{F}_{n,c}^0\left(\hat{X}_{j-1}^0\right) - \left(\hat{X}_j^0 + \hat{X}_{j-1}^0\right)\right\} + \frac{1}{3}n \tag{2.17}$$

with $\hat{X}_0^0 = 1$, $\hat{X}_{n+1}^0 = 1$.

Liao and Shimokawa (1999) proposed a new test statistic for a Weibull distribution with estimated parameters for a complete data set. Their new statistic is to combine the idea of the Kolmogorov-Smirnov, Cramér-von Mises, and Anderson-Darling statistics. If we generalize their new statistic to randomly censored data, it becomes

$$\hat{L}_{n,c} = \frac{1}{\sqrt{n}} \sum_{j=1,\delta_j=1}^{n} \frac{\max\left\{\hat{p}_{j,c} - \hat{F}^0\left(Y_j^0\right), \hat{F}^0\left(Y_j^0\right) - \hat{p}_{j-1,c}\right\}}{\sqrt{\hat{F}^0\left(Y_j^0\right)\left(1 - \hat{F}^0\left(Y_j^0\right)\right)}}.$$

This can also be written as

$$\hat{L}_{n,c} = \frac{1}{\sqrt{n}} \sum_{j=1,\delta_j=1}^{n} \frac{\max\left\{\hat{p}_{j,c} - \hat{X}_j^0, \hat{X}_j^0 - \hat{p}_{j-1,c}\right\}}{\sqrt{\hat{X}_j^0\left(1 - \hat{X}_j^0\right)}} \tag{2.18}$$

by (2.15). We will use $c = 0.5$ for the value of c in the next section since it is the most commonly used. Liao and Shimokawa (1999) also mentioned that the value $c = 0.5$ gives high power for complete data.

3. Estimation of parameters

For the observed random pairs (Y_i, δ_i), $i = 1, \ldots, n$, with

$$Y_i = \min\left(Y_i^0, T_i\right) \quad \text{and} \quad \delta_i = \begin{cases} 1, & \text{if } Y_i = Y_i^0, \\ 0, & \text{if } Y_i = T_i, \end{cases}$$

the log-likelihood function l is

$$l \propto \sum_{j=1}^{n} \delta_j \left[\ln\alpha - \alpha\ln\theta + (\alpha-1)\ln y_j - \left(\frac{y_j}{\theta}\right)^{\alpha} \right] - \sum_{j=1}^{n} \left(1-\delta_j\right)\left(\frac{y_j}{\theta}\right)^{\alpha},$$

when Y_i^0's follow a Weibull distribution with distribution function and pdf in (2.10) and (2.11). We assume that the distribution of the censoring time T_i's does not involve any parameters of interest. The assumption might be incorrect, but it is usually difficult to have some information for the censoring distribution.

The MLEs of α and θ can be derived by solving the likelihood equations

$$\frac{\partial l}{\partial\theta} = -\frac{n_u\alpha}{\theta} + \frac{\alpha}{\theta^{\alpha+1}} \sum_{j=1}^{n} y_j^{\alpha} = 0, \tag{3.1}$$

$$\frac{\partial l}{\partial\alpha} = \frac{n_u}{\alpha} - n_u\ln\theta + \sum_{j=1}^{n}\delta_j\ln y_j - \sum_{j=1}^{n}\left(\frac{y_j}{\theta}\right)^{\alpha}\ln\left(\frac{y_j}{\theta}\right) = 0, \tag{3.2}$$

where $n_u = \sum_{j=1}^{n}\delta_j$ is the number of uncensored data. From (3.1), we get

$$\theta(\alpha) = \left(\frac{\sum_{j=1}^{n} y_j^{\alpha}}{n_u}\right)^{\frac{1}{\alpha}}. \tag{3.3}$$

Substituting back $\theta(\alpha)$ into (3.2), we obtain the equation of α as

$$\frac{1}{\alpha} = \frac{\sum_{j=1}^{n} y_j^{\alpha}\ln y_j}{\sum_{j=1}^{n} y_j^{\alpha}} - \frac{1}{n_u}\sum_{j=1}^{n}\delta_j\ln y_j. \tag{3.4}$$

Balakrishnan and Kateri (2008) discussed the existence and uniqueness of the MLEs $\hat{\theta}, \hat{\alpha}$ of Weibull distributions that satisfy the equations (3.3), (3.4). It should be solved by an iterative way since the equation (3.4) does not have a solution in a compact form. Kim (2016), Pareek *et al.* (2009), and Kundu (2007) studied the approximate MLEs for parameters of Weibull distributions under several censoring schemes.

Another estimation method is to use graphical plotting method. The general concept of probability plotting is described in D'Agostino (1986). Graphical methods for survival distribution fitting are in Lee and Wang (2003). For Weibull plotting, we use type I extreme value distributions. By taking the logarithm of the distribution function of $EV(\mu,\sigma)$ in (2.12), we have a linear equation

$$v = \mu + \sigma\ln\left(-\ln\left(1 - F_E^0(v;\mu,\sigma)\right)\right).$$

The plot of the ordered value $v_i = \ln y_i$ versus $\ln(-\ln(1-p_i))$ for a plotting position p_i should be approximately on a straight line if the type I extreme value distribution reasonably fits the data. Liao and Shimokawa (1999) used the graphical plotting method to estimate the parameters of a Weibull distribution or a type I extreme value distribution on a complete data set and considered goodness of fit test statistics of distributions with estimated parameters.

We plot only uncensored data for randomly censored data. The plot therefore should be

$$\left(z_i = \ln\left(-\ln\left(1-\hat{p}_{i,c}\right)\right),\ v_i = \ln y_i\right), \quad \text{for } \delta_i = 1,$$

for the $\hat{p}_{i,c}$ defined in (2.6). By use of the least squares method, we can estimate the parameters σ and μ by

$$\tilde{\sigma} = \frac{\sum_{\delta_j=1}(z_j - \bar{z})(v_j - \bar{v})}{\sum_{\delta_j=1}(z_j - \bar{z})^2}, \tag{3.5}$$

$$\tilde{\mu} = \bar{v} - \tilde{\sigma}\bar{z}, \tag{3.6}$$

where $\bar{z} = \sum_{\delta_j=1} z_j/n_u$, $\bar{v} = \sum_{\delta_j=1} v_j/n_u$, $n_u = \sum_{j=1}^{n} \delta_j$. The parameters θ and α can be estimated by

$$\tilde{\theta} = \exp(\tilde{\mu}), \qquad \tilde{\alpha} = \frac{1}{\tilde{\sigma}} \tag{3.7}$$

according to (2.14). The graphical plotting method for estimation is easy to implement; however, it gives a bigger mean squared error than the MLEs.

4. Simulation study and example

4.1. Simulation results

A simulation study is conducted to give the null distributions and compare the power of the test statistics. The statistics $\hat{D}_{n,c}$, $\hat{\psi}^2_{n,c}$, and $\hat{L}_{n,c}$ in (2.16), (2.17), and (2.18) use the MLEs $\hat{\theta}$, $\hat{\alpha}$ in (3.3), (3.4). We write $\tilde{D}_{n,c}$, $\tilde{\psi}^2_{n,c}$, and $\tilde{L}_{n,c}$ when we use the graphical plotting method with the least squares to estimate θ and α. The estimators are given in (3.7). The upper percentage points of the statistics are given in Tables 1–3 for sample sizes $n = 20, 30, 40, 50, 100$, censored ratio $r = 0.2, 0.4, 0.5, 0.6$, and the significance level $\alpha = 0.01, 0.025, 0.05, 0.10, 0.15, 0.25, 0.50$. The values are based on random samples $N = 10{,}000$.

Similar to Liao and Shimokawa (1999), we can show that the test statistics are not dependent on the true values of θ and σ when the parameters are estimated by the MLEs or the graphical plotting method described in Section 3. Hence all the statistics are calculated when $\theta = 1$, $\sigma = 1$ without loss of generality.

We use the random censorship model proposed in Koziol and Green (1976) to control the censored ratio. It is

$$1 - G = (1 - F)^{\beta}, \quad \text{for some } \beta > 0, \tag{4.1}$$

where G is the distribution function of the censoring times, and β is called the censoring parameter. Under this model,

$$P\left(Y_i^0 > T_i\right) = \int_{-\infty}^{\infty} (1 - F(x))\, dG(x) = \int_0^1 \beta(1 - x)^{\beta} dx = \frac{\beta}{\beta + 1}.$$

In Tables 1–3, the ratio of the censored data r is equal to $\beta/(\beta + 1)$, which is the expected proportion of the censored observations. The motivation and characterization of this model is discussed in Csörgő and Horváth (1981), Chen *et al.* (1982), and Kim (2011, 2012).

The model in (4.1) does not satisfy the assumption that the distribution of the censoring time does not involve any parameters of interest. MLEs in Section 3 are derived ignoring the censoring model. However, we could not control the ratio of the censored simulated data without a censoring model, and the model is usually unknown in real situation. Kim (2016) investigated the influence of the assumption of the censoring model on the estimation of the parameters of a Weibull distribution.

Table 1: Upper tail percentage points for the test statistics $\sqrt{n}\hat{D}_{n,c}$, $\sqrt{n}\tilde{D}_{n,c}$ with unknown parameters, where r is the ratio of censored data

Statistic	n	r	α						
			0.01	0.025	0.05	0.10	0.15	0.25	0.5
$\sqrt{n}\hat{D}_{n,c}$	20	0.6	3.33	3.01	2.73	2.38	2.14	1.81	1.34
		0.5	2.67	2.34	2.08	1.79	1.61	1.38	1.08
		0.4	2.05	1.76	1.56	1.34	1.23	1.09	0.89
		0.2	1.20	1.10	1.02	0.94	0.89	0.81	0.70
	30	0.6	3.48	3.12	2.81	2.47	2.23	1.88	1.40
		0.5	2.71	2.30	2.05	1.77	1.61	1.38	1.08
		0.4	1.96	1.70	1.51	1.31	1.21	1.07	0.88
		0.2	1.17	1.09	1.01	0.93	0.87	0.80	0.69
	40	0.6	3.60	3.20	2.92	2.53	2.28	1.94	1.43
		0.5	2.76	2.41	2.13	1.82	1.63	1.40	1.09
		0.4	1.94	1.68	1.49	1.31	1.20	1.07	0.89
		0.2	1.14	1.06	0.98	0.91	0.86	0.79	0.68
	50	0.6	3.75	3.36	3.00	2.61	2.35	2.00	1.48
		0.5	2.71	2.40	2.11	1.82	1.63	1.41	1.11
		0.4	1.89	1.61	1.44	1.26	1.17	1.05	0.88
		0.2	1.14	1.06	0.98	0.90	0.85	0.78	0.68
	100	0.6	3.98	3.60	3.21	2.80	2.52	2.16	1.60
		0.5	2.70	2.40	2.11	1.84	1.65	1.43	1.13
		0.4	1.81	1.57	1.40	1.24	1.15	1.04	0.88
		0.2	1.12	1.03	0.97	0.89	0.84	0.78	0.67
$\sqrt{n}\tilde{D}_{n,c}$	20	0.6	3.33	3.01	2.76	2.43	2.23	1.93	1.48
		0.5	2.70	2.39	2.15	1.90	1.73	1.52	1.20
		0.4	2.13	1.90	1.70	1.50	1.37	1.21	0.98
		0.2	1.33	1.22	1.13	1.03	0.97	0.89	0.76
	30	0.6	3.48	3.14	2.86	2.54	2.33	2.03	1.55
		0.5	2.75	2.41	2.16	1.92	1.75	1.54	1.21
		0.4	2.10	1.87	1.68	1.48	1.37	1.22	0.98
		0.2	1.33	1.21	1.13	1.04	0.98	0.89	0.76
	40	0.6	3.63	3.26	2.98	2.63	2.41	2.09	1.60
		0.5	2.82	2.55	2.29	1.98	1.81	1.57	1.24
		0.4	2.09	1.87	1.68	1.50	1.38	1.23	1.00
		0.2	1.31	1.20	1.11	1.01	0.96	0.87	0.75
	50	0.6	3.76	3.41	3.08	2.71	2.49	2.15	1.65
		0.5	2.79	2.50	2.25	1.99	1.83	1.60	1.26
		0.4	2.05	1.84	1.65	1.47	1.36	1.21	0.99
		0.2	1.31	1.19	1.11	1.02	0.96	0.87	0.75
	100	0.6	4.03	3.65	3.30	2.91	2.67	2.34	1.80
		0.5	2.80	2.54	2.30	2.04	1.88	1.66	1.31
		0.4	2.06	1.83	1.68	1.50	1.39	1.24	1.00
		0.2	1.29	1.18	1.10	1.01	0.95	0.87	0.73

The values of $\sqrt{n}\hat{D}_{n,c}$ and $\sqrt{n}\tilde{D}_{n,c}$ in Table 1 are very similar. For the other two statistics, the values are bigger for the graphical plotting method, and the differences are significantly bigger for the Liao-Shimokawa statistic. Note that the goodness-of-fit tests are not distribution free when the parameters should be estimated. Kim (2012) studied the Kolmogorov-Smirnov statistic D_n in (2.7) to test exponentiality for randomly censored data when the scale parameter is unknown. The values for $\sqrt{n}\hat{D}_{n,c}$, $\sqrt{n}\tilde{D}_{n,c}$ in Table 1 and the values in Kim (2012) multiplied by $\sqrt{n}$ show just a little difference. The estimation procedure does not seem to significantly change the null distribution of the Kolmogorov-Smirnov statistic. However, the null distribution of the Liao-Shimokawa statistic seems strongly influenced by the parameter estimation.

Table 2: Upper tail percentage points for the test statistics $\hat{\psi}_{n,c}$, $\tilde{\psi}_{n,c}$ with unknown parameters, where r is the ratio of censored data

Statistic	n	r	α 0.01	0.025	0.05	0.10	0.15	0.25	0.5
$\hat{\psi}_{n,c}$	20	0.6	2.77	2.03	1.54	1.05	0.78	0.52	0.27
		0.5	1.43	1.00	0.71	0.48	0.38	0.28	0.18
		0.4	0.69	0.47	0.36	0.26	0.22	0.18	0.12
		0.2	0.23	0.19	0.17	0.14	0.12	0.11	0.08
	30	0.6	2.64	1.89	1.42	0.98	0.75	0.49	0.27
		0.5	1.24	0.84	0.61	0.44	0.34	0.26	0.17
		0.4	0.51	0.37	0.29	0.23	0.20	0.16	0.11
		0.2	0.22	0.18	0.16	0.13	0.12	0.10	0.07
	40	0.6	2.57	1.87	1.37	0.92	0.71	0.49	0.27
		0.5	1.15	0.81	0.58	0.39	0.32	0.24	0.16
		0.4	0.49	0.35	0.28	0.22	0.19	0.15	0.11
		0.2	0.21	0.18	0.15	0.13	0.11	0.09	0.07
	50	0.6	2.58	1.83	1.36	0.93	0.71	0.49	0.27
		0.5	0.98	0.70	0.53	0.37	0.30	0.23	0.15
		0.4	0.43	0.32	0.26	0.21	0.18	0.14	0.11
		0.2	0.21	0.18	0.15	0.13	0.11	0.09	0.07
	100	0.6	2.22	1.61	1.21	0.84	0.67	0.47	0.28
		0.5	0.76	0.56	0.44	0.33	0.27	0.22	0.15
		0.4	0.33	0.27	0.22	0.18	0.16	0.14	0.10
		0.2	0.20	0.17	0.14	0.12	0.10	0.09	0.06
$\tilde{\psi}_{n,c}$	20	0.6	2.79	2.08	1.64	1.21	0.95	0.67	0.37
		0.5	1.60	1.13	0.87	0.64	0.52	0.39	0.24
		0.4	0.84	0.62	0.49	0.37	0.31	0.24	0.16
		0.2	0.31	0.26	0.22	0.18	0.16	0.13	0.09
	30	0.6	2.75	2.08	1.64	1.20	0.94	0.68	0.38
		0.5	1.39	1.04	0.82	0.60	0.50	0.37	0.23
		0.4	0.76	0.57	0.44	0.34	0.29	0.23	0.15
		0.2	0.30	0.25	0.21	0.17	0.15	0.13	0.09
	40	0.6	2.72	2.06	1.59	1.18	0.94	0.69	0.39
		0.5	1.40	1.03	0.80	0.59	0.48	0.36	0.22
		0.4	0.71	0.56	0.44	0.34	0.28	0.22	0.15
		0.2	0.31	0.25	0.21	0.17	0.15	0.12	0.08
	50	0.6	2.75	2.07	1.62	1.20	0.96	0.69	0.39
		0.5	1.27	0.95	0.75	0.57	0.47	0.36	0.22
		0.4	0.68	0.52	0.41	0.32	0.27	0.22	0.14
		0.2	0.30	0.25	0.21	0.17	0.15	0.12	0.08
	100	0.6	2.52	1.93	1.55	1.18	0.96	0.70	0.41
		0.5	1.08	0.85	0.69	0.53	0.44	0.34	0.21
		0.4	0.61	0.49	0.40	0.31	0.26	0.21	0.14
		0.2	0.30	0.25	0.21	0.17	0.14	0.12	0.08

The asymptotic null distributions of the statistics remain unknown. The asymptotic distribution of ψ_n^2 in (2.9) is studied by Koziol and Green (1976) only when the null hypothesis is fully specified with known parameters. The numbers in Table 1 and Table 2 indicate that the null distributions of the Kolmogorov-Smirnov and Koziol-green statistics converge rapidly when the ratio of the censored data is small. We can find the p-value of each goodness-of-fit test approximately by using the points in Tables 1–3.

Next, we investigate the power of the test statistics. Table 4 and Table 5 present the power of the statistics at the significance level $\alpha = 0.10$ for sample sizes $n = 50, 100$. We generated $N = 5{,}000$ samples for each alternative. The alternatives are listed below.

Table 3: Upper tail percentage points for the test statistics $\hat{L}_{n,c}$, $\tilde{L}_{n,c}$ with unknown parameters, where r is the ratio of censored data

Statistic	n	r	α						
			0.01	0.025	0.05	0.10	0.15	0.25	0.5
$\hat{L}_{n,c}$	20	0.6	0.80	0.73	0.68	0.62	0.58	0.53	0.42
		0.5	0.90	0.82	0.76	0.70	0.67	0.61	0.51
		0.4	0.97	0.89	0.83	0.77	0.73	0.68	0.59
		0.2	1.14	1.05	0.98	0.91	0.87	0.82	0.73
	30	0.6	0.72	0.66	0.61	0.56	0.52	0.48	0.40
		0.5	0.82	0.75	0.70	0.64	0.61	0.56	0.48
		0.4	0.91	0.83	0.77	0.72	0.68	0.63	0.55
		0.2	1.03	0.96	0.91	0.84	0.80	0.75	0.67
	40	0.6	0.67	0.62	0.57	0.53	0.50	0.45	0.39
		0.5	0.76	0.71	0.66	0.61	0.58	0.53	0.46
		0.4	0.86	0.79	0.73	0.68	0.65	0.60	0.53
		0.2	1.02	0.94	0.87	0.81	0.77	0.72	0.64
	50	0.6	0.66	0.59	0.55	0.51	0.48	0.44	0.38
		0.5	0.74	0.68	0.63	0.58	0.55	0.51	0.44
		0.4	0.82	0.75	0.70	0.65	0.62	0.58	0.51
		0.2	0.99	0.91	0.85	0.79	0.75	0.70	0.62
	100	0.6	0.56	0.53	0.49	0.45	0.43	0.39	0.34
		0.5	0.66	0.61	0.57	0.53	0.50	0.46	0.40
		0.4	0.75	0.69	0.65	0.60	0.57	0.53	0.46
		0.2	0.92	0.84	0.79	0.73	0.69	0.64	0.56
$\tilde{L}_{n,c}$	20	0.6	1.22	0.96	0.82	0.71	0.66	0.59	0.49
		0.5	1.62	1.16	0.96	0.82	0.76	0.68	0.57
		0.4	2.05	1.35	1.09	0.93	0.84	0.76	0.64
		0.2	2.84	1.73	1.36	1.14	1.03	0.91	0.77
	30	0.6	1.16	0.88	0.75	0.65	0.60	0.55	0.46
		0.5	1.45	1.10	0.91	0.77	0.70	0.63	0.54
		0.4	1.63	1.20	1.02	0.87	0.80	0.71	0.60
		0.2	2.30	1.50	1.23	1.04	0.95	0.84	0.72
	40	0.6	1.01	0.82	0.71	0.63	0.58	0.52	0.44
		0.5	1.26	0.97	0.85	0.74	0.67	0.61	0.52
		0.4	1.58	1.15	0.95	0.82	0.76	0.68	0.58
		0.2	1.98	1.40	1.15	0.99	0.90	0.81	0.68
	50	0.6	0.97	0.78	0.69	0.60	0.56	0.50	0.43
		0.5	1.12	0.93	0.81	0.70	0.65	0.58	0.50
		0.4	1.34	1.06	0.91	0.79	0.73	0.65	0.56
		0.2	1.63	1.30	1.11	0.96	0.88	0.79	0.67
	100	0.6	0.79	0.69	0.61	0.54	0.51	0.46	0.38
		0.5	0.92	0.79	0.71	0.63	0.59	0.53	0.45
		0.4	1.08	0.93	0.82	0.72	0.67	0.60	0.51
		0.2	1.30	1.11	0.98	0.87	0.80	0.72	0.61

- Gamma distribution, Gamma(α), $\alpha = 0.5, 2$, with pdf $f(t;\alpha) = t^{\alpha-1}e^{-t}/\Gamma(\alpha),\ \ t > 0$.
- Lognormal distribution, lognorm(σ), $\sigma = 0.5, 1, 2$, with $f(t;\sigma) = 1/(\sqrt{2\pi}\sigma t)e^{-(\log t)^2/2\sigma^2},\ \ t > 0$.
- Log-logistic distribution, log-logistic(σ), $\sigma = 0.5, 1, 2$, with $f(t;\sigma) = t^{1/\sigma-1}/(\sigma(1 + t^{1/\sigma})^2)$, $t > 0$.
- Log double exponential, log-DE, DE with pdf $f(t) = e^{-|x|}/2$.
- Half-logistic with $f(t) = 2e^{-t}/(1 + e^{-t})^2$, $t > 0$.

The power results indicate the following. The meaningfully good power is written in bold in each

Table 4: Power comparison of $\hat{D}_{n,c}$, $\hat{\psi}^2_{n,c}$, $\hat{L}_{n,c}$, $\tilde{D}_{n,c}$, $\tilde{\psi}^2_{n,c}$, and $\tilde{L}_{n,c}$ for $\alpha = 0.10$ and $n = 50$

Distribution	Censoring ratio	$\hat{D}_{n,c}$	$\hat{\psi}^2_{n,c}$	$\hat{L}_{n,c}$	$\tilde{D}_{n,c}$	$\tilde{\psi}^2_{n,c}$	$\tilde{L}_{n,c}$
Weibull	$r = 0.6$	0.10	0.09	0.10	0.10	0.10	0.10
	$r = 0.5$	0.10	0.10	0.09	0.09	0.10	0.10
	$r = 0.4$	0.11	0.10	0.10	0.11	0.10	0.10
	$r = 0.2$	0.11	0.10	0.10	0.10	0.10	0.10
Gamma(0.5)	$r = 0.6$	0.10	0.10	0.10	0.10	0.08	0.08
	$r = 0.5$	0.10	0.09	0.11	0.08	0.08	0.06
	$r = 0.4$	0.10	0.12	0.13	0.09	0.09	0.07
	$r = 0.2$	0.17	0.18	0.16	0.13	0.15	0.07
Gamma(2)	$r = 0.6$	0.10	0.10	0.11	0.12	0.12	**0.15**
	$r = 0.5$	0.10	0.12	0.12	0.13	0.13	**0.17**
	$r = 0.4$	0.12	0.12	0.13	0.14	0.15	**0.19**
	$r = 0.2$	0.09	0.11	0.14	0.13	0.13	**0.22**
Lognorm(0.5)	$r = 0.6$	0.11	0.11	0.25	0.16	0.20	**0.41**
	$r = 0.5$	0.13	0.20	0.32	0.22	0.28	**0.50**
	$r = 0.4$	0.18	0.28	0.41	0.32	0.36	**0.61**
	$r = 0.2$	0.26	0.41	0.53	0.35	0.43	**0.77**
Lognorm(1)	$r = 0.6$	0.10	0.11	0.25	0.15	0.19	**0.39**
	$r = 0.5$	0.12	0.17	0.33	0.24	0.30	**0.50**
	$r = 0.4$	0.18	0.27	0.40	0.33	0.36	**0.62**
	$r = 0.2$	0.27	0.41	0.54	0.34	0.41	**0.77**
Lognorm(2)	$r = 0.6$	0.11	0.11	0.25	0.15	0.19	**0.40**
	$r = 0.5$	0.13	0.18	0.32	0.22	0.29	**0.50**
	$r = 0.4$	0.19	0.27	0.40	0.32	0.35	**0.60**
	$r = 0.2$	0.26	0.41	0.54	0.34	0.41	**0.76**
Log-logistic(0.5)	$r = 0.6$	0.11	0.11	0.20	0.14	0.14	**0.25**
	$r = 0.5$	0.13	0.18	0.27	0.17	0.20	**0.35**
	$r = 0.4$	0.20	0.29	0.39	0.24	0.25	**0.48**
	$r = 0.2$	0.36	0.48	0.61	0.26	0.31	**0.69**
Log-logistic(1)	$r = 0.6$	0.11	0.11	0.18	0.13	0.14	**0.25**
	$r = 0.5$	0.12	0.17	0.28	0.17	0.20	**0.35**
	$r = 0.4$	0.19	0.28	0.39	0.24	0.24	**0.48**
	$r = 0.2$	0.35	0.48	0.60	0.28	0.31	**0.69**
Log-logistic(2)	$r = 0.6$	0.11	0.11	0.20	0.14	0.15	**0.24**
	$r = 0.5$	0.13	0.19	0.28	0.17	0.19	**0.35**
	$r = 0.4$	0.18	0.28	0.38	0.24	0.25	**0.47**
	$r = 0.2$	0.36	0.48	0.60	0.26	0.30	**0.69**
Log-double exponential	$r = 0.6$	0.10	0.11	**0.18**	0.11	0.09	0.14
	$r = 0.5$	0.13	0.20	**0.28**	0.12	0.12	0.24
	$r = 0.4$	0.26	0.37	**0.45**	0.17	0.16	0.38
	$r = 0.2$	0.58	0.71	**0.76**	0.31	0.39	0.71
Half-logistic	$r = 0.6$	0.10	0.09	0.10	0.09	0.08	0.07
	$r = 0.5$	0.09	0.10	0.11	0.09	0.09	0.07
	$r = 0.4$	0.11	0.10	0.12	0.09	0.08	0.07
	$r = 0.2$	0.13	0.12	0.12	0.10	0.12	0.07

alternative. First, $\tilde{L}_{n,c}$ shows the highest power for Gamma(2), lognormal, and log-logistic alternatives. $\hat{L}_{n,c}$ has the second highest power for lognormal and log-logistic distributions. According to Liao and Shimokawa (1999), $\tilde{L}_{n,c}$ was also the most powerful for a complete sample. $\hat{L}_{n,c}$ has the best power, and $\tilde{L}_{n,c}$ has the second best power for a log-DE alternative. All statistics show a poor power against gamma and half-logistic alternatives.

Second, the power in decreasing order seem to be judged as $\hat{L}_{n,c} > \hat{\psi}^2_{n,c} > \hat{D}_{n,c}$ when the parameters are estimated with the MLEs. When they are coupled with a graphical plotting method, $\tilde{L}_{n,c}$ has the

Table 5: Power comparison of $\hat{D}_{n,c}$, $\hat{\psi}^2_{n,c}$, $\hat{L}_{n,c}$, $\tilde{D}_{n,c}$, $\tilde{\psi}^2_{n,c}$, and $\tilde{L}_{n,c}$ for $\alpha = 0.10$ and $n = 100$

Distribution	Censoring ratio	$\hat{D}_{n,c}$	$\hat{\psi}^2_{n,c}$	$\hat{L}_{n,c}$	$\tilde{D}_{n,c}$	$\tilde{\psi}^2_{n,c}$	$\tilde{L}_{n,c}$
Weibull	$r = 0.6$	0.10	0.11	0.10	0.11	0.10	0.10
	$r = 0.5$	0.09	0.09	0.10	0.10	0.11	0.10
	$r = 0.4$	0.11	0.10	0.10	0.10	0.11	0.10
	$r = 0.2$	0.09	0.09	0.09	0.10	0.11	0.09
Gamma(0.5)	$r = 0.6$	0.10	0.11	0.13	0.10	0.07	0.09
	$r = 0.5$	0.10	0.10	0.14	0.10	0.10	0.10
	$r = 0.4$	0.11	0.15	0.16	0.10	0.12	0.10
	$r = 0.2$	0.22	0.25	0.22	0.19	0.24	0.13
Gamma(2)	$r = 0.6$	0.11	0.11	0.13	0.12	0.14	**0.18**
	$r = 0.5$	0.11	0.14	0.15	0.15	0.18	**0.21**
	$r = 0.4$	0.13	0.14	0.16	0.17	0.19	**0.24**
	$r = 0.2$	0.11	0.16	0.19	0.17	0.18	**0.28**
Lognorm(0.5)	$r = 0.6$	0.12	0.15	0.44	0.26	0.37	**0.62**
	$r = 0.5$	0.15	0.33	0.55	0.40	0.56	**0.76**
	$r = 0.4$	0.28	0.51	0.66	0.54	0.65	**0.84**
	$r = 0.2$	0.53	0.72	0.82	0.65	0.75	**0.94**
Lognorm(1)	$r = 0.6$	0.12	0.16	0.43	0.27	0.36	**0.61**
	$r = 0.5$	0.16	0.31	0.57	0.40	0.57	**0.76**
	$r = 0.4$	0.28	0.52	0.66	0.55	0.65	**0.84**
Lognorm(2)	$r = 0.6$	0.12	0.16	0.44	0.24	0.37	**0.62**
	$r = 0.5$	0.16	0.31	0.55	0.41	0.56	**0.76**
	$r = 0.4$	0.28	0.52	0.67	0.55	0.65	**0.85**
	$r = 0.2$	0.55	0.72	0.82	0.64	0.75	**0.95**
Log-logistic(0.5)	$r = 0.6$	0.12	0.14	0.31	0.17	0.21	**0.34**
	$r = 0.5$	0.15	0.30	0.46	0.27	0.34	**0.50**
	$r = 0.4$	0.31	0.52	0.62	0.36	0.40	**0.66**
	$r = 0.2$	0.66	0.79	0.86	0.50	0.58	**0.89**
Log-logistic(1)	$r = 0.6$	0.12	0.16	0.31	0.17	0.21	**0.35**
	$r = 0.5$	0.16	0.29	0.46	0.27	0.33	**0.51**
	$r = 0.4$	0.31	0.52	0.62	0.36	0.41	**0.65**
	$r = 0.2$	0.67	0.80	0.85	0.50	0.57	**0.89**
Log-logistic(2)	$r = 0.6$	0.11	0.14	0.30	0.17	0.22	**0.34**
	$r = 0.5$	0.16	0.29	0.46	0.28	0.34	**0.50**
	$r = 0.4$	0.32	0.53	0.62	0.37	0.40	**0.66**
	$r = 0.2$	0.66	0.80	0.85	0.51	0.58	**0.89**
Log-double exponential	$r = 0.6$	0.10	0.16	**0.27**	0.12	0.11	0.19
	$r = 0.5$	0.20	0.37	**0.49**	0.17	0.15	0.35
	$r = 0.4$	0.48	0.68	**0.72**	0.25	0.28	0.55
	$r = 0.2$	0.89	0.94	**0.95**	0.61	0.68	0.90
Half-logistic	$r = 0.6$	0.11	0.10	0.12	0.09	0.08	0.09
	$r = 0.5$	0.10	0.10	0.12	0.09	0.08	0.09
	$r = 0.4$	0.10	0.11	0.12	0.09	0.10	0.09
	$r = 0.2$	0.14	0.15	0.14	0.13	0.15	0.10

highest power, and $\tilde{\psi}^2_{n,c}$ has comparable to or a little higher power than $\tilde{D}_{n,c}$. Third, the power depends on how the parameters are estimated. But the MLEs or the graphical plotting method does not give consistently better power. The estimators having better power are different from each alternative for each statistic. Lastly, the power of lognormal or log-logistic alternatives does not depend on the parameter σ, just like the null distributions of the statistics.

4.2. Example

As an illustrative example for goodness-of-fit testing, we take the remission times of 21 patients with acute leukemia who received 6-mercaptopurine (6-MP). The study was to assess the ability of the 6-MP treatment to maintain remission. We randomized 42 patients to receive 6-MP or a placebo. The study was ended in one year. The data were originally reported by Freireich *et al.* (1963) in a clinical trial. It is also discussed in Lee and Wang (2003) and Kleinbaum and Klein (2005). The recorded data are remission times, in weeks, of the 6-MP. The placebo group is not shown.

6	6	6	7	10	13	16	22	23			
6+	9+	10+	11+	17+	19+	20+	25+	32+	32+	34+	35+

Lee and Wang (2003) checked the Weibull hazard plot for the given data and showed the Weibull distribution provided a good fit.

When we compute the statistics proposed in this paper for the remission data above, we have

$$\sqrt{n}\hat{D}_{n,c} = 2.18 \ (p\text{-value} \approx 0.15), \qquad \sqrt{n}\tilde{D}_{n,c} = 2.18 \ (0.15 < p\text{-value} < 0.25)$$
$$\hat{\psi}^2_{n,c} = 0.77 \ (p\text{-value} \approx 0.15), \qquad \tilde{\psi}^2_{n,c} = 0.94 \ (0.15 < p\text{-value} < 0.25)$$
$$\hat{L}_{n,c} = 0.33 \ (p\text{-value} > 0.5), \qquad \tilde{L}_{n,c} = 0.55 \ (0.25 < p\text{-value} < 0.5).$$

The p-values or the range of p-values are found approximately using the percentile points of $n = 20$, $r = 0.6$ in each first line of Tables 1–3. Each statistic gives a different p-value; however, it supports a Weibull distribution can not be rejected.

5. Concluding remarks

In this paper, the Kolmogorov-Smirnov, Koziol-Green, and Liao-Shimokawa statistics are applied to test randomly censored Weibull distributions with estimated parameters. Statistics are generalized to randomly censored cases using the Kaplan-Meier product limit of the distribution function just like Koziol and Green (1976).

The upper percentage points of the statistics are provided by simulations. The parameters are estimated by the MLEs and the graphical plotting method. The null distributions depend on the estimation method since the test statistics are not distribution free when the parameters are estimated. Through the simulation study, the Liao-Shimokawa statistic showed relatively good power among competitors. However the null distribution of the statistic changes too much upon the parameter estimation. Hence we do not recommend the use of the statistic simply because of the good power. We instead recommend the Koziol and Green statistic since it shows slightly better power than the Kolmogorov-Smirnov; in addition, the null distribution does not heavily depend on the estimation procedure.

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