On the maximum likelihood estimators for parameters of a Weibull distribution under random censoring

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Abstract

In this paper, we consider statistical inferences on the estimation of the parameters of a Weibull distribution when data are randomly censored. Maximum likelihood estimators (MLEs) and approximate MLEs are derived to estimate the parameters. We consider two cases for the censoring model: the assumption that the censoring distribution does not involve any parameters of interest and a censoring distribution that follows a Weibull distribution. A simulation study is conducted to compare the performances of the estimators. The result shows that the MLEs and the approximate MLEs are similar in terms of biases and mean square errors; in addition, the assumption of the censoring model has a strong influence on the estimation of scale parameter.

Keywords: approximate maximum likelihood estimators, Koziol-Green model, maximum likelihood estimators, power comparison, random censoring, Weibull distribution

1. Introduction

In survival studies, data may be subject to random censoring. Let $T_1, \dots, T_n$ denote lifetimes with distribution function F and probability density function (pdf) f . The censoring times $C_1, \dots, C_n$ drawn independently of the T_i are from distribution function G and pdf g . The T_i 's are censored on the right by C_i , so we observe n i.i.d. random pairs (X_j, δ_j) , $j = 1, \dots, n$, where

$$X_j = \min(T_j, C_j) \quad \text{and} \quad \delta_j = \begin{cases} 1, & \text{if } X_j = T_j, \\ 0, & \text{if } X_j = C_j. \end{cases} \quad (1.1)$$

For a model of random censorship, Koziol and Green (1976) introduced a special model with

$$1 - G = (1 - F)^\beta, \quad \text{for some } \beta > 0. \quad (1.2)$$

Under this model, we have

$$P(T_i > C_i) = \int_{-\infty}^{\infty} (1 - F(x)) dG(x) = \int_0^1 \beta(1 - x)^\beta dx = \frac{\beta}{\beta + 1}.$$

Hence, $\beta/(\beta + 1)$ is the expected proportion of the censored observations and β is called the censoring parameter. The case $\beta = 0$ corresponds to no censoring. Csörgő and Horváth (1981) called (1.2) the Koziol-Green model of random censorship. The motivation and characterization of this model is discussed in Chen *et al.* (1982) and Kim (2014).

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Suppose the lifetime distribution follows a Weibull distribution $\text{Weibull}(\lambda, \alpha)$ with the distribution function

$$F(x) = 1 - e^{-\lambda x^\alpha}, \quad x > 0, \quad (1.3)$$

and the pdf

$$f(x) = \alpha \lambda x^{\alpha-1} e^{-\lambda x^\alpha}, \quad x > 0, \quad (1.4)$$

where λ and α are the scale and shape parameters, respectively, and are positive.

This paper derives the maximum likelihood estimators (MLEs) and the approximate MLEs of the parameters of a Weibull distribution under random censoring. The analysis of censored data usually assumes that the censoring distribution does not involve any parameters of interest. We find the MLEs and the approximate MLEs in this case and those under the Koziol-Green model in (1.2) to compare them. In Section 2, we obtain the MLEs of the parameters of a Weibull distribution. We also derive the approximate MLEs using the type I extreme value distribution of the minimum since the MLEs cannot be expressed in explicit form. In Section 3, we conduct a simulation study to compare the MLEs and the approximate MLEs. We generate censored random samples under the Koziol-Green model and compute the estimators in two different ways. One is the estimators ignoring the Koziol-Green model (assuming the censoring distribution does not involve any parameters) and the other is the estimators under the true model. We compare the two of them to see what happens when we ignore the random censorship model. Section 4 ends the paper with some concluding remarks.

2. MLEs and the approximate MLEs

2.1. MLEs and the approximate MLEs under the assumption that the censoring distribution does not involve any parameters

For the observed random pairs (X_j, δ_j) in (1.1), we assume that X_j 's are the ordered observations without loss of generality. The likelihood function based on (X_j, δ_j) is given by (Tableman and Kim, 2003)

$$\begin{aligned} L &= n! \prod_{i=1}^n \left(f(x_i) \bar{G}(x_i) \right)^{\delta_i} \left(g(x_i) \bar{F}(x_i) \right)^{1-\delta_i} \\ &= n! \prod_{i=1}^n f(x_i)^{\delta_i} \bar{F}(x_i)^{1-\delta_i} \prod_{i=1}^n g(x_i)^{1-\delta_i} \bar{G}(x_i)^{\delta_i}, \end{aligned} \quad (2.1)$$

where $\bar{F} = 1 - F$, $\bar{G} = 1 - G$. If we assume that the distribution of the censoring time C_i 's does not involve any parameters of interest, the part $\prod_{i=1}^n g(x_i)^{1-\delta_i} \bar{G}(x_i)^{\delta_i}$ in (2.1) plays no role in the estimation process. Hence the likelihood function can be taken to be

$$L \propto \prod_{i=1}^n f(x_i)^{\delta_i} \bar{F}(x_i)^{1-\delta_i}.$$

If the lifetime distribution follows $\text{Weibull}(\lambda, \alpha)$ given in (1.4) or (1.3), then the log-likelihood function $l = \ln L$ becomes

$$l \propto \sum_{i=1}^n \delta_i \left[\ln \alpha + \ln \lambda + (\alpha - 1) \ln x_i - \lambda x_i^\alpha \right] + \sum_{i=1}^n (1 - \delta_i) (-\lambda x_i^\alpha).$$

The MLEs of α and λ can be derived by solving the likelihood equations

$$\frac{\partial l}{\partial \lambda} = \frac{n_u}{\lambda} - \sum_{i=1}^n x_i^\alpha = 0, \quad (2.2)$$

$$\frac{\partial l}{\partial \alpha} = \frac{n_u}{\alpha} + \sum_{i=1}^n \delta_i \ln x_i - \lambda \sum_{i=1}^n x_i^\alpha \ln x_i = 0, \quad (2.3)$$

where $n_u = \sum_{i=1}^n \delta_i$ is the number of uncensored data. From (2.2), we get

$$\lambda(\alpha) = \frac{n_u}{\sum_{i=1}^n x_i^\alpha}. \quad (2.4)$$

Substituting back $\lambda(\alpha)$ into (2.3), we obtain the equation of α as

$$\frac{1}{\alpha} = \frac{\sum_{i=1}^n x_i^\alpha \ln x_i}{\sum_{i=1}^n x_i^\alpha} - \frac{1}{n_u} \sum_{i=1}^n \delta_i \ln x_i. \quad (2.5)$$

Balakrishnan and Kateri (2008) discussed the existence and uniqueness of the MLEs of a Weibull distribution. The equation (2.5) does not have a solution in a compact form and it should be solved in an iterative way.

If the random variable T follows $\text{Weibull}(\alpha, \lambda)$, then $\ln T$ follows the type I extreme value distribution of the minimum $\text{EV}(\mu, \sigma)$ with the distribution function and the pdf,

$$F_Y(y) = 1 - e^{-e^{\frac{y-\mu}{\sigma}}}, \quad f_Y(y) = \frac{1}{\sigma} e^{\frac{y-\mu}{\sigma} - e^{\frac{y-\mu}{\sigma}}}, \quad -\infty < y < \infty, \quad (2.6)$$

where μ and σ are location and scale parameters respectively, and $\mu = -(1/\alpha) \ln \lambda$, $\sigma = 1/\alpha$. For the estimation of the extreme value distribution, we refer to Chen (1998) and Engelhardt and Bain (1974).

Pareek *et al.* (2009) mentioned that $\text{Weibull}(\alpha, \lambda)$ and $\text{EV}(\mu, \sigma)$ are equivalent models because the procedure developed under one model can be used for the other. Nonetheless, we could have an easier estimation process by working with the model $\text{EV}(\mu, \sigma)$ in (2.6) since μ and σ are the location and scale parameters. A similar technique was used by Kundu (2007) and Kang and Han (2009) for different censoring schemes.

Letting $Y_j = \ln X_j$, the log-likelihood function based on (Y_j, δ_j) becomes

$$l \propto \sum_{i=1}^n \delta_i \left[-\ln \sigma + \frac{y_i - \mu}{\sigma} - e^{\frac{y_i - \mu}{\sigma}} \right] + \sum_{i=1}^n (1 - \delta_i) \left(-e^{\frac{y_i - \mu}{\sigma}} \right). \quad (2.7)$$

Remind that $Z = (Y - \mu)/\sigma$ has a standard extreme value distribution $\text{EV}(0, 1)$ with the distribution function and the pdf

$$F_Z(z) = 1 - e^{-e^z}, \quad f_Z(z) = e^{z - e^z}$$

when Y follows $\text{EV}(\mu, \sigma)$. Substituting $z_i = (y_i - \mu)/\sigma$ in (2.7), we get

$$l(\mu, \sigma) \propto n_u (-\ln \sigma) + \sum_{i=1}^n \delta_i z_i - \sum_{i=1}^n e^{z_i}. \quad (2.8)$$

Since $\partial z_i / \partial \mu = -(1/\sigma)$, $\partial e^{z_i} / \partial \mu = -(1/\sigma)e^{z_i}$, $\partial z_i / \partial \sigma = -(1/\sigma)z_i$, and $\partial e^{z_i} / \partial \sigma = -(1/\sigma)z_i e^{z_i}$, we have

$$\begin{aligned}\frac{\partial l}{\partial \mu} &= -\frac{n_u}{\sigma} + \frac{1}{\sigma} \sum_{i=1}^n e^{z_i} = 0, \\ \frac{\partial l}{\partial \sigma} &= -\frac{n_u}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^n \delta_i z_i + \frac{1}{\sigma} \sum_{i=1}^n z_i e^{z_i} = 0,\end{aligned}$$

by taking derivatives with respect to μ and σ of $l(\mu, \sigma)$ in (2.8). Equivalently

$$n_u - \sum_{i=1}^n e^{z_i} = 0, \quad (2.9)$$

$$n_u + \sum_{i=1}^n \delta_i z_i - \sum_{i=1}^n z_i e^{z_i} = 0. \quad (2.10)$$

Clearly, (2.9) and (2.10) do not have explicit solutions. By expanding $h(z_j) = e^{z_j}$ in Taylor series keeping only the first 2 terms around

$$\xi_j = F_Z^{-1}(p_j) = \ln(-\ln q_j), \quad q_j = 1 - p_j, \quad (2.11)$$

for some p_j defined later, we approximate

$$h(z_j) = e^{z_j} \cong h(\xi_j) + h'(\xi_j)(z_j - \xi_j) \equiv a_j + b_j z_j \quad (2.12)$$

with $a_j = (-\ln q_j)(1 - \ln(-\ln q_j))$, $b_j = -\ln q_j$. Substituting (2.12) and $z_j = (y_j - \mu)/\sigma$ into (2.9) and (2.10), we have

$$\sigma \left[\sum_{i=1}^n a_i - n_u \right] + \sum_{i=1}^n b_i y_i = \mu \sum_{i=1}^n b_i, \quad (2.13)$$

$$\sigma^2 n_u + \sigma \left[\sum_{i=1}^n \delta_i (y_i - \mu) - \sum_{i=1}^n a_i (y_i - \mu) \right] - \sum_{i=1}^n b_i (y_i - \mu)^2 = 0. \quad (2.14)$$

From (2.13), we obtain the approximate solution $\tilde{\mu}$ as

$$\tilde{\mu} = A + B\tilde{\sigma} \quad (2.15)$$

with $A = \sum_{i=1}^n b_i y_i / \sum_{i=1}^n b_i$ and $B = (\sum_{i=1}^n a_i - n_u) / \sum_{i=1}^n b_i$. Substituting (2.15) into (2.14), we get the quadratic equation

$$C\sigma^2 - D\sigma - E = 0 \quad \text{or} \quad n_u\sigma^2 - D\sigma - E = 0, \quad (2.16)$$

where

$$\begin{aligned}C &= n_u - B \left[n_u - \sum_{i=1}^n a_i + B \sum_{i=1}^n b_i \right] = n_u, \\ D &= \sum_{i=1}^n a_i (y_i - A) - 2B \sum_{i=1}^n b_i (y_i - A) - \sum_{i=1}^n \delta_i (y_i - A), \\ E &= \sum_{i=1}^n b_i (y_i - A)^2 > 0.\end{aligned}$$

Since $b_j = -\ln q_j > 0$ and $E > 0$,

$$\tilde{\sigma} = \frac{D + \sqrt{D^2 + 4n_u E}}{2n_u}$$

is the only positive root of the quadratic equation (2.16), and it can be the approximate MLE of σ . The approximate MLEs of α and λ can be $\tilde{\alpha} = 1/\tilde{\sigma}$ and $\tilde{\lambda} = \lambda(\tilde{\alpha})$ from (2.4).

Let us think about the plotting position p_j in (2.11). For randomly censored data, we can use the Kaplan-Meier estimator

$$p_j^{KM} = 1 - \prod_{i \leq j} \left(\frac{n-i}{n-i+1} \right)^{\delta_i},$$

which has been studied in Kaplan and Meier (1958), Efron (1967), Breslow and Crowley (1974), and Meier (1975). The modified one is

$$p_j^{(1)} = 1 - \prod_{i \leq j} \left(\frac{n-i+1}{n-i+2} \right)^{\delta_i} \left(\frac{n}{n+1} \right)^{1-\delta_1}. \quad (2.17)$$

We put the part $(n/(n+1))^{1-\delta_1}$ on the similar quantile probabilities proposed by Herd (1960) and Johnson (1964) to avoid 0 values. Another simple choice is the usual Weibull plotting position $p_j^{(2)} = j/(n+1)$. Note that $p_j^{(1)}$ reduce to $p_j^{(2)} = j/(n+1)$ for a complete sample. We call $\tilde{\sigma}_1, \tilde{\alpha}_1, \tilde{\lambda}_1$ for the approximate MLEs when we use $p_j^{(1)}$ in (2.17) and $\tilde{\sigma}_2, \tilde{\alpha}_2, \tilde{\lambda}_2$ when $p_j^{(2)} = j/(n+1)$ is used.

2.2. MLEs and the approximate MLEs under the Koziol-Green model

If we assume the Koziol-Green model (1.2) in the case that the lifetime distribution follows Weibull(λ, α), the censoring distribution also follows a Weibull distribution with the same shape parameter α . Let us assume G follows Weibull(η, α), then we have the censoring parameter $\beta = \eta/\lambda$ and the expected proportion of the censored data $\eta/(\lambda + \eta)$. The likelihood function in (2.1) still gives the MLE of λ as in (2.4) and the equation of α as

$$\frac{1}{\alpha} = \frac{\sum_{i=1}^n x_i^\alpha \ln x_i}{\sum_{i=1}^n x_i^\alpha} - \frac{1}{n} \sum_{i=1}^n \ln x_i, \quad (2.18)$$

which is also the likelihood equation of α for a complete sample. We can obtain the likelihood equation (2.18) from the fact that the distribution of $X_j = \min(T_j, C_j)$ follows Weibull($\lambda + \eta, \alpha$).

As for the approximate MLE of α , we use the extreme value distribution EV(μ, σ) in (2.6) with $\mu = -(1/\alpha) \ln(\lambda + \eta)$, $\sigma = 1/\alpha$. The same approach in Section 2.1 gives

$$\tilde{\mu} = A + B\tilde{\sigma}$$

and the quadratic equation in σ

$$n\sigma^2 - D\sigma - E = 0, \quad (2.19)$$

Table 1: Averages of the MLE $\hat{\alpha}$ and the approximate MLEs $\tilde{\alpha}_1, \tilde{\alpha}_2$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 2, \lambda = 0.4, 0.6, 0.8$ under the assumption that the censoring distribution does not involve any parameters

λ	n	$\hat{\alpha}$	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	MSE($\hat{\alpha}$)	MSE($\tilde{\alpha}_1$)	MSE($\tilde{\alpha}_2$)	mean($\hat{\alpha} - \tilde{\alpha}_1$)	mean($\hat{\alpha} - \tilde{\alpha}_2$)	RE($\tilde{\alpha}_1, \hat{\alpha}$)	RE($\tilde{\alpha}_2, \hat{\alpha}$)
0.4	20	2.285	2.129	2.034	0.666	0.375	0.238	0.155	0.251	56.243	35.738
	30	2.153	2.108	2.023	0.316	0.242	0.151	0.045	0.130	76.709	47.707
	40	2.110	2.062	1.997	0.225	0.169	0.105	0.047	0.112	75.040	46.785
	50	2.092	2.074	2.012	0.152	0.145	0.086	0.018	0.079	95.276	56.290
0.6	20	2.193	2.136	2.079	0.345	0.287	0.224	0.056	0.114	83.332	64.858
	30	2.141	2.110	2.066	0.208	0.173	0.139	0.031	0.075	83.006	66.802
	40	2.095	2.078	2.041	0.133	0.123	0.099	0.017	0.054	92.247	74.165
	50	2.088	2.079	2.047	0.106	0.098	0.078	0.009	0.041	92.162	73.708
0.8	20	2.168	2.146	2.121	0.234	0.214	0.196	0.023	0.047	91.435	83.964
	30	2.102	2.091	2.073	0.136	0.130	0.121	0.010	0.029	95.579	88.635
	40	2.080	2.076	2.061	0.100	0.097	0.091	0.004	0.019	97.166	90.474
	50	2.071	2.062	2.049	0.077	0.077	0.072	0.009	0.022	99.128	93.077

MLE = maximum likelihood estimator, RE = relative efficiency, MSE = mean squared error.

where

$$\begin{aligned}
 A &= \frac{\sum_{i=1}^n b_i y_i}{\sum_{i=1}^n b_i}, \\
 B &= \frac{\sum_{i=1}^n a_i - n}{\sum_{i=1}^n b_i}, \\
 D &= \sum_{i=1}^n a_i (y_i - A) - 2B \sum_{i=1}^n b_i (y_i - A) - \sum_{i=1}^n (y_i - A), \\
 E &= \sum_{i=1}^n b_i (y_i - A)^2 > 0,
 \end{aligned}$$

and $a_j = (-\ln q_j)(1 - \ln(-\ln q_j))$, $b_j = -\ln q_j$, $q_j = 1 - p_j$, $p_j = j/(n+1)$. Therefore the approximate MLE of σ becomes the positive root of the equation (2.19),

$$\tilde{\sigma} = \frac{D + \sqrt{D^2 + 4nE}}{2n}$$

and $\tilde{\alpha} = 1/\tilde{\sigma}$.

3. Simulation results

In this section, a simulation study is conducted to compare the performance of the MLEs with the approximate MLEs in terms of biases and mean squared errors (MSEs) under the different assumptions of the censoring model. Random samples are generated when the true model is the Koziol-Green model in (1.2). We use the sample sizes $n = 20, 30, 40, 50$ and the parameter values $\alpha = 2, 0.5$, $(\lambda, \eta) = (0.4, 0.6), (0.6, 0.4), (0.8, 0.2)$, which give the proportion of the censoring data 0.6, 0.4, 0.2, respectively. Tables 1–4 provide averages of the MLEs and the approximate MLEs, their MSEs, differences between them, and relative efficiency (RE) for the parameters α and λ , respectively, under the assumption that the censoring distribution does not involve any parameters. Tables 5–8 provide the same statistics under the Koziol-Green model. We replicate each process 2,000 times. The RE,

Table 2: Averages of the MLE $\hat{\lambda}$ and the approximate MLEs $\tilde{\lambda}_1, \tilde{\lambda}_2$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 2, \lambda = 0.4, 0.6, 0.8$ under the assumption that the censoring distribution does not involve any parameters

λ	n	$\hat{\lambda}$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	MSE($\hat{\lambda}$)	MSE($\tilde{\lambda}_1$)	MSE($\tilde{\lambda}_2$)	mean($\hat{\lambda} - \tilde{\lambda}_1$)	mean($\hat{\lambda} - \tilde{\lambda}_2$)	RE($\tilde{\lambda}_1, \hat{\lambda}$)	RE($\tilde{\lambda}_2, \hat{\lambda}$)
0.4	20	0.407	0.416	0.424	0.026	0.025	0.024	-0.010	-0.017	95.312	92.540
	30	0.403	0.406	0.414	0.017	0.017	0.016	-0.003	-0.011	97.566	94.282
	40	0.404	0.407	0.413	0.012	0.011	0.011	-0.003	-0.009	97.588	94.681
	50	0.399	0.400	0.406	0.010	0.009	0.009	-0.001	-0.007	99.598	95.153
0.6	20	0.616	0.621	0.628	0.042	0.041	0.040	-0.006	-0.012	97.853	95.309
	30	0.609	0.612	0.618	0.025	0.024	0.024	-0.003	-0.009	97.631	95.252
	40	0.611	0.613	0.618	0.019	0.019	0.018	-0.002	-0.007	98.324	96.440
	50	0.607	0.608	0.612	0.015	0.015	0.015	-0.001	-0.005	99.375	97.387
0.8	20	0.827	0.830	0.833	0.108	0.108	0.108	-0.002	-0.006	99.854	100.396
	30	0.814	0.815	0.818	0.081	0.081	0.082	-0.001	-0.004	99.926	100.799
	40	0.813	0.813	0.816	0.069	0.070	0.070	0.000	-0.003	100.201	101.267
	50	0.808	0.809	0.811	0.062	0.062	0.063	-0.001	-0.003	100.837	101.928

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

Table 3: Averages of the MLE $\hat{\alpha}$ and the approximate MLEs $\tilde{\alpha}_1, \tilde{\alpha}_2$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 0.5, \lambda = 0.4, 0.6, 0.8$ under the assumption that the censoring distribution does not involve any parameters

λ	n	$\hat{\alpha}$	$\tilde{\alpha}_1$	$\tilde{\alpha}_2$	MSE($\hat{\alpha}$)	MSE($\tilde{\alpha}_1$)	MSE($\tilde{\alpha}_2$)	mean($\hat{\alpha} - \tilde{\alpha}_1$)	mean($\hat{\alpha} - \tilde{\alpha}_2$)	RE($\tilde{\alpha}_1, \hat{\alpha}$)	RE($\tilde{\alpha}_2, \hat{\alpha}$)
0.4	20	0.560	0.528	0.503	0.040	0.023	0.014	0.032	0.057	56.973	34.912
	30	0.545	0.526	0.506	0.021	0.015	0.010	0.019	0.038	72.217	45.068
	40	0.527	0.520	0.502	0.013	0.012	0.007	0.007	0.024	86.643	52.821
	50	0.522	0.515	0.500	0.010	0.009	0.005	0.008	0.023	85.355	52.461
0.6	20	0.549	0.535	0.521	0.023	0.017	0.014	0.014	0.028	77.586	60.953
	30	0.529	0.521	0.510	0.012	0.010	0.008	0.008	0.019	84.453	67.114
	40	0.521	0.519	0.509	0.008	0.008	0.006	0.002	0.011	90.331	72.481
	50	0.514	0.513	0.505	0.006	0.006	0.005	0.001	0.010	99.539	80.320
0.8	20	0.539	0.535	0.529	0.015	0.015	0.013	0.004	0.010	98.553	90.588
	30	0.528	0.526	0.521	0.008	0.008	0.008	0.002	0.007	101.966	94.120
	40	0.519	0.516	0.513	0.006	0.006	0.005	0.002	0.006	94.039	87.951
	50	0.513	0.512	0.509	0.005	0.004	0.004	0.001	0.004	97.638	91.554

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

for example, between two estimates $\tilde{\alpha}, \hat{\alpha}$ is defined as

$$\text{RE}(\tilde{\alpha}, \hat{\alpha}) = \frac{\text{MSE}(\tilde{\alpha})}{\text{MSE}(\hat{\alpha})} \times 100.$$

From Tables 1–4, where the estimation process is conducted under the assumption that the censoring distribution does not involve any parameters, we observe the following. First, the approximate MLEs $\tilde{\alpha}_1, \tilde{\lambda}_1$ using the modified Kaplan-Meier estimator (2.17) give the closer values to the MLEs $\hat{\alpha}, \hat{\lambda}$ on average than the approximate MLEs $\tilde{\alpha}_2, \tilde{\lambda}_2$ with $p_j^{(2)} = j/(n+1)$ for all censoring ratios and sample sizes considered. This phenomena happen for both parameters α and λ . Second, the estimators $\tilde{\alpha}_2, \tilde{\lambda}_2$ have the smallest MSEs, and the MLEs $\hat{\alpha}, \hat{\lambda}$ have the largest MSEs for almost all cases. The MSEs are almost the same as for parameter λ , and usually the MSEs of the MLE $\hat{\alpha}$ are significantly bigger than the MSEs of the approximate MLEs as for parameter α . We can also see that the REs between the approximate MLEs and the MLEs tend to decrease as the censoring ratio increases (λ decreases) or the sample size decrease.

Table 4: Averages of the MLE $\hat{\lambda}$ and the approximate MLEs $\tilde{\lambda}_1, \tilde{\lambda}_2$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 0.5, \lambda = 0.4, 0.6, 0.8$ under the assumption that the censoring distribution does not involve any parameters

λ	n	$\hat{\lambda}$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	MSE($\hat{\lambda}$)	MSE($\tilde{\lambda}_1$)	MSE($\tilde{\lambda}_2$)	mean($\hat{\lambda} - \tilde{\lambda}_1$)	mean($\hat{\lambda} - \tilde{\lambda}_2$)	RE($\tilde{\lambda}_1, \hat{\lambda}$)	RE($\tilde{\lambda}_2, \hat{\lambda}$)
0.4	20	0.404	0.411	0.419	0.027	0.026	0.025	-0.007	-0.015	96.387	93.325
	30	0.402	0.407	0.414	0.017	0.016	0.016	-0.006	-0.012	93.121	90.685
	40	0.404	0.405	0.412	0.013	0.012	0.012	-0.002	-0.008	97.581	93.924
	50	0.405	0.408	0.414	0.010	0.009	0.009	-0.003	-0.008	95.671	93.205
0.6	20	0.616	0.622	0.628	0.089	0.089	0.091	-0.005	-0.012	100.767	102.439
	30	0.610	0.614	0.620	0.070	0.071	0.073	-0.004	-0.010	101.851	104.216
	40	0.608	0.609	0.614	0.062	0.062	0.064	0.000	-0.005	99.876	102.249
	50	0.605	0.606	0.610	0.056	0.056	0.058	-0.001	-0.005	100.045	102.423
0.8	20	0.838	0.840	0.843	0.249	0.251	0.253	-0.002	-0.005	100.629	101.450
	30	0.816	0.817	0.820	0.208	0.209	0.211	-0.001	-0.004	100.506	101.394
	40	0.813	0.814	0.816	0.195	0.196	0.198	-0.001	-0.004	100.410	101.269
	50	0.814	0.814	0.816	0.190	0.191	0.192	0.000	-0.003	100.053	100.850

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

Table 5: Averages of the MLE $\hat{\alpha}$ and the approximate MLE $\tilde{\alpha}$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 2, \lambda = 0.4, 0.6, 0.8$ under the Koziol-Green model

λ	n	$\hat{\alpha}$	$\tilde{\alpha}$	MSE($\hat{\alpha}$)	MSE($\tilde{\alpha}$)	mean($\hat{\alpha} - \tilde{\alpha}$)	RE($\tilde{\alpha}, \hat{\alpha}$)
0.4	20	2.152	2.143	0.196	0.192	0.008	97.737
	30	2.103	2.100	0.114	0.113	0.003	98.783
	40	2.065	2.063	0.076	0.076	0.002	99.039
	50	2.064	2.063	0.059	0.059	0.001	100.314
0.6	20	2.143	2.135	0.178	0.175	0.008	98.278
	30	2.112	2.107	0.113	0.111	0.004	98.351
	40	2.075	2.073	0.076	0.076	0.002	99.309
	50	2.075	2.074	0.062	0.062	0.001	100.179
0.8	20	2.154	2.146	0.181	0.177	0.008	97.821
	30	2.096	2.092	0.108	0.107	0.004	99.238
	40	2.078	2.076	0.079	0.079	0.002	99.509
	50	2.063	2.062	0.063	0.063	0.001	100.080

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

Table 6: Averages of the MLE $\hat{\lambda}$ and the approximate MLEs $\tilde{\lambda}$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 2, \lambda = 0.4, 0.6, 0.8$ under the Koziol-Green model

λ	n	$\hat{\lambda}$	$\tilde{\lambda}$	MSE($\hat{\lambda}$)	MSE($\tilde{\lambda}$)	mean($\hat{\lambda} - \tilde{\lambda}$)	RE($\tilde{\lambda}, \hat{\lambda}$)
0.4	20	0.418	0.419	0.024	0.024	-0.0003	99.791
	30	0.410	0.410	0.016	0.016	-0.0001	99.830
	40	0.408	0.409	0.011	0.011	-0.0001	99.971
	50	0.402	0.402	0.009	0.009	0.0000	100.001
0.6	20	0.623	0.624	0.041	0.041	-0.0005	99.776
	30	0.614	0.615	0.024	0.024	-0.0003	99.850
	40	0.615	0.615	0.018	0.018	-0.0001	99.881
	50	0.609	0.609	0.015	0.015	0.0000	99.970
0.8	20	0.831	0.831	0.108	0.108	-0.0006	100.030
	30	0.816	0.816	0.081	0.081	-0.0004	100.123
	40	0.814	0.814	0.070	0.070	-0.0002	100.077
	50	0.810	0.810	0.062	0.062	0.0000	100.020

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

Table 7: Averages of the MLE $\hat{\alpha}$ and the approximate MLE $\tilde{\alpha}$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 0.5$, $\lambda = 0.4, 0.6, 0.8$ under the Koziol-Green model

λ	n	$\hat{\alpha}$	$\tilde{\alpha}$	MSE($\hat{\alpha}$)	MSE($\tilde{\alpha}$)	mean($\hat{\alpha} - \tilde{\alpha}$)	RE($\tilde{\alpha}, \hat{\alpha}$)
0.4	20	0.533	0.531	0.011	0.011	0.002	99.037
	30	0.525	0.524	0.007	0.007	0.001	98.778
	40	0.518	0.518	0.005	0.005	0.000	99.569
	50	0.513	0.513	0.003	0.003	0.000	99.698
0.6	20	0.538	0.536	0.011	0.011	0.002	98.230
	30	0.521	0.520	0.006	0.006	0.001	99.288
	40	0.517	0.516	0.005	0.005	0.000	99.520
	50	0.511	0.511	0.004	0.004	0.000	100.309
0.8	20	0.536	0.534	0.012	0.012	0.002	98.801
	30	0.526	0.525	0.007	0.007	0.001	98.819
	40	0.517	0.516	0.005	0.005	0.001	99.561
	50	0.512	0.512	0.004	0.004	0.000	99.591

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

Table 8: Averages of the MLE $\hat{\lambda}$ and the approximate MLEs $\tilde{\lambda}$, MSEs, differences between the MLE and the approximate MLE and RE of two estimates when $\alpha = 0.5$, $\lambda = 0.4, 0.6, 0.8$ under the Koziol-Green model

λ	n	$\hat{\lambda}$	$\tilde{\lambda}$	MSE($\hat{\lambda}$)	MSE($\tilde{\lambda}$)	mean($\hat{\lambda} - \tilde{\lambda}$)	RE($\tilde{\lambda}, \hat{\lambda}$)
0.4	20	0.413	0.414	0.025	0.025	-0.0004	99.877
	30	0.410	0.410	0.016	0.016	-0.0002	99.951
	40	0.408	0.408	0.012	0.012	-0.0001	100.029
	50	0.410	0.410	0.009	0.009	0.0000	99.977
0.6	20	0.623	0.624	0.090	0.090	-0.0004	100.069
	30	0.616	0.616	0.072	0.072	-0.0003	100.077
	40	0.611	0.611	0.063	0.063	-0.0001	100.071
	50	0.608	0.608	0.057	0.057	0.0000	100.006
0.8	20	0.841	0.842	0.251	0.252	-0.0007	100.168
	30	0.818	0.819	0.210	0.210	-0.0004	100.114
	40	0.814	0.815	0.196	0.196	-0.0002	100.078
	50	0.815	0.815	0.191	0.191	-0.0001	100.028

MLE = maximum likelihood estimator, MSE = mean squared error, RE = relative efficiency.

From Tables 5–8, where the estimation is conducted under the true model (the Koziol-Green model in (1.2)), the average differences between the MLEs and the approximate MLEs are negligible. Apparently the MSEs depend on sample sizes rather than censoring ratios. The assumption of the censoring distribution has a strong influence on the estimation of α rather than that of λ , especially when the censoring ratio is big.

4. Concluding remarks

In this paper, we study randomly censored data from a Weibull distribution. We derive the approximate MLEs in explicit forms using the type I extreme value distribution of the minimum since the MLEs of the parameters do not have explicit solutions. In the simulation study, we generate random samples under the Koziol-Green model, compute the estimators ignoring the censoring model, and compare them with the estimators from the true model. Consequently, we find that the censoring model has a strong influence on the scale parameter α rather than the shape parameter λ . The MSE of the scale parameter becomes larger when we ignore the censoring model. The approximate MLEs are quite similar to the MLEs in terms of biases and MSEs in any situation.

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