Asymptotic computation of Greeks under a stochastic volatility model

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Abstract

We study asymptotic expansion formulae for numerical computation of Greeks (i.e. sensitivity) in finance. Our approach is based on the integration-by-parts formula of the Malliavin calculus. We propose asymptotic expansion of Greeks for a stochastic volatility model using the Greeks formula of the Black-Scholes model. A singular perturbation method is applied to derive asymptotic Greeks formulae. We also provide numerical simulation of our method and compare it to the Monte Carlo finite difference approach.

Keywords: computation of Greeks, asymptotics, stochastic volatility, singular perturbation, Malliavin calculus

1. Introduction

Greeks are quantities representing partial derivatives of the price of a contingent claim. The Greeks are also called (risk) sensitivities since they describe movements of the price of a contingent claim with respect to risks. For example, the delta measures a rate of change of the option value with respect to the price change of the underlying asset (the first partial derivatives with respect to the underlying), and the vega is the derivative of the option value with respect to the volatility of the underlying asset. Greeks are very useful for derivatives traders, especially those who want to build a hedging portfolio. However, it is hard to obtain an explicit formula of a Greek in general financial models, including a stochastic volatility model.

Monte Carlo simulation for the finite difference approximation is a common approach to find Greeks. For instance, each sample of the Delta is estimated by $(1/\epsilon)(P(t, x + \epsilon) - P(t, x))$ for a small ϵ , where $P(t, x)$ is the price of the derivative at time t and with the underlying value x . According to the Glasserman and Yao (1992) and Glynn (1989), the Monte Carlo simulation for the Greeks has a bad convergence rate and performs poorly if the payoff function is not smooth enough.

Another approach to calculate Greeks is the Malliavin calculus approach, which is introduced in Fournié *et al.* (1999, 2001). This approach, which we call the Malliavin Greeks, works well even when the payoff function is not continuous, such as a digital option.

However, the Malliavin Greeks approach requires the first variation process of the underlying to obtain Greeks. In general, the first variation process is given by the solution of the stochastic differential equation (SDE) associated with the underlying process. A few models such as Black-Scholes (B.S.) have an explicit formula of their first variation process. The choice of the optimal weight is also an important issue in the Malliavin Greeks. Fournié *et al.* (1999, 2001) show that

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an explicit optimal weight of the Black-Scholes model is easy to calculate; therefore, we can obtain efficient Malliavin Greeks under the Black-Scholes model.

A stochastic volatility model has been a popular alternative of the Black Scholes model. In a stochastic volatility model, the volatility is not a constant but another stochastic process and can describe various uncertain movements of volatility. Therefore, the stochastic volatility model became popular in finance. A typical stochastic volatility model assumes that the volatility is driven by a mean-reverting process. Hull and White (1987), Heston (1993), Fouque *et al.* (2000, 2003, 2011) are well known studies on stochastic volatility models. In particular, Fouque *et al.* (2000, 2003, 2011) derived an asymptotic solution for European options under a fast mean reverting stochastic volatility setting after observing a fast time scale volatility factor of mean reversion.

In this paper, we apply the Malliavin Greeks approach in Fournié *et al.* (1999, 2001) for an asymptotic expansion of the stochastic volatility model developed by Fouque *et al.* (2000, 2003, 2011). We derive Greeks formulae of a stochastic volatility model using the Black-Scholes model. The singular perturbation method and the Malliavin calculus are two main tools. We compare the finite difference approximation of Greeks to our results by numerical simulations.

Recently, Takahashi and Yamada (2012, 2014) studied the asymptotic expansions of stochastic volatility model. They present asymptotic expansions of option prices and Greeks that provided an error estimates around the Black-Scholes model in stochastic volatility model. The basic difference between our study and theirs is the underlying model. The stochastic volatility model developed by Fouque *et al.* (2000, 2003, 2011) is a singular problem. Precisely, the volatility process of this model are diverged as the parameter of model goes to zero. However, Takahashi and Yamada (2012, 2014) studied regular cases, i.e. the volatility process are converged to a positive constant as the parameter of model goes to zero. Therefore, their approaches cannot be applied to the stochastic volatility model developed by Fouque *et al.* (2000, 2003, 2011).

The rest of the paper is organized as follows. In the Section 2, we briefly review the Malliavin calculus and results in Fournié *et al.* (1999, 2001). Section 3 presents asymptotic Greeks under the fast mean reversion model developed by Fouque *et al.* (2003). Numerical examples are provided in Section 4 that numerically verify the results in Section 3.

2. Malliavin calculus and a computation of Greeks

This section gives a brief review of the Malliavin calculus and the main results of Fournié *et al.* (1999, 2001). The Malliavin calculus introduces derivatives of the Wiener functional and can be presented as integration by part. We refer to Fournié *et al.* (1999, 2001) for details.

Let $\{W_t, 0 \leq t \leq T\}$ be an n -dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $\mathcal{F}_t$ generated by W . Let F be a random variable of the form

$$F = f\left(\int_0^T h_1(t)dW_t, \dots, \int_0^T h_n(t)dW_t\right), \quad (2.1)$$

where f is an infinitely differentiable and rapidly decreasing function on $\mathbb{R}^n$, and $h_i \in L^2([0, T] \times \Omega)$. The Malliavin derivative DF is defined as the process $\{D_t F, t \geq 0\}$ of $L^2([0, T] \times \Omega)$

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left(\int_0^T h_1(t)dW_t, \dots, \int_0^T h_n(t)dW_t \right) h_i(t), \quad t \geq 0 \text{ a.s.} \quad (2.2)$$

We also define the norm for F by

$$\|F\|_{1,2} = \left(E[F^2]\right)^{\frac{1}{2}} + \left(E\left[\int_0^T (D_t F)^2 dt\right]\right)^{\frac{1}{2}}. \quad (2.3)$$

Let $\mathbb{D}^{1,2}$ denote the Banach space which is the completion of the set of F with respect to the norm $\|\cdot\|_{1,2}$. Then D is a closed linear operator from $\mathbb{D}^{1,2}$ to $L^2([0, T] \times \Omega)$. We now present some useful results in Fournié *et al.* (1999, 2001).

Proposition 1. *Let $\phi : R^n \rightarrow R$ be a continuously differentiable function with bounded derivatives, and F be a random vector in $\mathbb{D}^{1,2}$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and*

$$D_t \phi(F) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(F) D_t F_i, \quad t \geq 0 \text{ a.s.} \quad (2.4)$$

Proposition 1 presents a chain rule of Malliavin derivatives. The next proposition provides Malliavin derivatives of the solution of SDEs.

Proposition 2. *Let $\{X_t, t \geq 0\}$ be an R^n valued process driven by the SDE*

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \quad (2.5)$$

where α and σ are continuously differentiable with bounded derivatives. Let $\{Y_t, t \geq 0\}$ be the associated first variation process defined by

$$dY_t = \alpha'(X_t)Y_t dt + \sum_{i=1}^n \sigma'_i(X_t)Y_t dW_t^i, \quad Y_0 = I_n, \quad (2.6)$$

where I_n is the identity matrix of R^n , primes denote derivatives, and σ_i is the i -th column vector of σ . We also assume the diffusion matrix σ satisfies

$$\xi^T \sigma^T(x) \sigma(x) \xi \geq c|\xi|^2, \quad \text{for any } \xi, x \in R^n, \quad (2.7)$$

for some $c > 0$. Then the process $\{X_t, t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by

$$D_s X_t = Y_t(Y_s)^{-1} \sigma(X_s) 1_{s \leq t}, \quad s \geq 0 \text{ a.s.} \quad (2.8)$$

Therefore, If $\phi \in C_b^1(R^n)$ then we have

$$D_s \phi(X_t) = \nabla \phi(X_t) Y_t(Y_s)^{-1} \sigma(X_s) 1_{s \leq t}, \quad s \geq 0 \text{ a.s.} \quad (2.9)$$

and

$$D_s \int_0^T \phi(X_t) dt = \int_s^T \nabla \phi(X_t) Y_t(Y_s)^{-1} \sigma(X_s) dt, \quad \text{a.s.} \quad (2.10)$$

We now review the Skorohod integral, which is associated with Malliavin derivatives.

Definition 1. *We say that $u \in \text{Dom}(\delta)$, if*

$$E \left[\int_0^T u_t D_t \phi dt \right] < \infty$$

for a stochastic process u and for any $\phi \in \mathbb{D}^{1,2}$. Let us define the Skorohod integral $\delta(u) := \int_0^T u_t \delta W_t$ by

$$E \left[\int_0^t u_t D_t \phi dt \right] = E [\phi \delta(u)], \quad \text{for all } \phi \in \mathbb{D}^{1,2}.$$

Actually, $\text{Dom}(\delta)$ contains $\mathcal{F}_t$ -adapted processes which belongs to $L^2([0, T] \times \Omega)$; In this case, the divergence operator δ coincides with Ito integral,

$$\int_0^T u_t \delta W_t = \int_0^T u_t dW_t. \quad (2.11)$$

Next proposition presents an integration by part formula for Malliavin derivatives.

Proposition 3. *Let F be a $\mathcal{F}_t$ -adapted random variable in $\mathbb{D}^{1,2}$. Then for any $u \in \text{Dom}(\delta)$ we have*

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u_t dt. \quad (2.12)$$

Proposition 4 is called the Clark-Ocone formula and that is very useful. It expresses the value of the function F as the sum of its mean value and an Itô integral with respect to Malliavin derivatives.

Proposition 4. *Let F be a random variable in $\mathbb{D}^{1,2}$. Then we have*

$$F = E[F] + \int_0^T E[D_t F | \mathcal{F}_t] dW_t. \quad (2.13)$$

Proposition 5. *For any fixed $t \in [0, T]$ and $u \in \text{Dom}(\delta)$, we obtain $1_{(0,t]} u \in \text{Dom}(\delta)$ and*

$$\int_0^t u_s \delta W_s = \int_0^T 1_{(0,t]} u_s \delta W_s, \quad (2.14)$$

$$\int_0^t u_s \delta W_s + \int_t^T u_s \delta W_s = \int_0^T u_s \delta W_s. \quad (2.15)$$

2.1. Greeks and Malliavin calculus

We assume that the underlying asset is the solution of the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \quad (2.16)$$

which is the special case when $n = 1$ in Proposition 2. We now consider a European option pricing problem under the risk neutral measure $\mathbb{Q}$

$$P_X(t, x) = E^{\mathbb{Q}} \left[e^{-r(T-t)} h(X_T) | X_t = x \right],$$

where the payoff function h belongs to $L^2(R)$. We use the notation $E^{X_t}[\cdot] := E^{\mathbb{Q}}[\cdot | X_t = x]$ for convenience.

Let us study a Malliavin Greeks formula of a European option. The next theorem presents the delta of a European option price.

Theorem 1. Let $a(\tau)$ be a continuous deterministic function. Suppose that

$$\int_t^T a(\tau) d\tau = 1,$$

for a fixed $T > t$. Then we have

$$\frac{\partial}{\partial x} (E^{X_t}[h(X_T)]) = E^{X_t}[h(X_T)\pi^\Delta], \quad (2.17)$$

$$\pi^\Delta := \int_t^T a(s) \frac{Y_s}{\sigma(X_s)} dW_s, \quad (2.18)$$

where Y_t is the first variation process of X_t .

Proof: Refer to the proof of the Proposition 3.2 in Fournié *et al.* (1999). $\square$

Remark 1. According to Fournié *et al.* (1999), the payoff function h does not need to be smooth to guarantee the efficient convergence rate of Malliavin Greeks. Using the same argument in Theorem 1, we can derive other Greeks such as Gamma(the second derivatives w.r.t. the price) and Vega(the derivatives w.r.t. σ). We refer to Fournié *et al.* (1999) for details.

3. Fast mean-reverting OU volatility model and singular perturbation method

In this section, we study stochastic volatility model with volatility driven by a fast mean-reverting Ornstein-Uhlenbeck process. The fast mean reverting stochastic volatility model is first developed by Fouque *et al.* (2003) and is an application of the asymptotic analysis in option pricing. This stochastic volatility model leads to an asymptotic option price by the ergodic theorem. Consequently, they derived an option price of a fast mean-reverting stochastic volatility model from the option price of the Black-Scholes model.

We study the same fast mean-reverting stochastic volatility model, and study how to calculate Greeks borrowing a similar idea of adopting the Black-Scholes model. Under a risk-neutral measure, the underlying price X_t of the fast mean reverting stochastic volatility model satisfies SDE

$$\begin{aligned} \frac{dX_t}{X_t} &= rdt + f(V_t)dW_t^{(0)}, \\ dV_t &= \left[\frac{1}{\epsilon}(m - V_t) - \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}\Lambda(V_t) \right] dt + \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}dW_t^{(1)}, \end{aligned} \quad (3.1)$$

where $W_t^{(0)}$ and $W_t^{(1)}$ are dependent one dimensional Brownian motions with correlated coefficient ρ , and $\Lambda(v)$ is given by

$$\Lambda(v) = \rho \frac{(\mu - r)}{f(v)} + \gamma(v) \sqrt{1 - \rho^2}. \quad (3.2)$$

Here, the market price of risk $\gamma(v)$ is determined by Girsanov theorem. Constants μ and ν^2 are the mean and the variance of the invariant distribution V_∞ , respectively. We assume that ϵ is small enough, which means that the volatility process $V(t)$ is fast mean-reverting.

Under this formulation, the option price

$$C(t, x, v) = E^{\mathbb{Q}} \left[e^{-r(T-t)} h(X_T) | X_t = x, V_t = v \right]$$

satisfies the partial differential equation (PDE),

$$\left[\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right] C = 0, \quad (3.3)$$

where

$$\begin{aligned} \mathcal{L}_0 &= (m - v) \frac{\partial}{\partial v} + v^2 \frac{\partial^2}{\partial v^2}, \\ \mathcal{L}_1 &= -\sqrt{2}v\Lambda(v) \frac{\partial}{\partial v} + \sqrt{2}v\rho f(v)x \frac{\partial^2}{\partial x \partial v}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} - r \bullet + rx \frac{\partial}{\partial x} + \frac{1}{2} f^2(v)x^2 \frac{\partial^2}{\partial x^2} \end{aligned}$$

with the terminal condition of $C(T, x, y) = h(x)$. Let $C = \sum_{n=0}^{\infty} \epsilon^{n/2} C_n$, then the PDE (3.3) becomes

$$\left[\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right] C = \frac{1}{\epsilon} [\mathcal{L}_0 C_0] + \frac{1}{\sqrt{\epsilon}} [\mathcal{L}_1 C_0 + \mathcal{L}_0 C_1] + [\mathcal{L}_2 C_0 + \mathcal{L}_0 C_2 + \mathcal{L}_1 C_1] + \cdots = 0.$$

By the solvability condition (refer to Fouque *et al.* (2003)), the leading order term C_0 and the first correction term C_1 are independent of v . Moreover, C_0 and C_1 satisfy partial differential equations

$$\begin{aligned} \langle \mathcal{L}_2 \rangle C_0(t, x) &= 0, \\ C_0(T, x) &= h(x) \end{aligned}$$

and

$$\begin{aligned} \langle \mathcal{L}_2 \rangle C_1(t, x) &= \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \right\rangle C_0 = -(A_1 - 2A_2)x^2 \frac{\partial^2 C_0}{\partial x^2} + A_2 x^3 \frac{\partial^3 C_0}{\partial x^3}, \\ C_1(T, x) &= 0, \end{aligned}$$

respectively. Here, $A_1 = (v/\sqrt{2})\langle \phi' \Lambda \rangle$ and $A_2 = (\rho v/\sqrt{2})\langle \phi' f \rangle$. ϕ is the solution of $\mathcal{L}_0 \phi(v) = f^2(v) - \langle f^2 \rangle$. The bracket $\langle \cdot \rangle$ denotes the expectation with respect to the invariant distribution of Y defined by

$$\langle g \rangle = \int g(v) \Phi(v) dv, \quad \text{and} \quad \Phi(v) := \frac{1}{\sqrt{2\pi}v^2} e^{-\frac{(v-m)^2}{2v^2}}.$$

Note that the operator $\langle \mathcal{L}_2 \rangle$ is exactly same to the operator $\mathcal{L}_2$ by replacing $f(v)$ with the constant $\bar{\sigma} = \sqrt{\langle f^2 \rangle}$.

We now discuss the Delta formula of $C(t, x, v)$ driven by Malliavin derivatives.

Theorem 2. *Let S_t be the solution of the following SDE:*

$$dS_t = rS_t dt + \bar{\sigma} S_t dW_t^{(0)}, \quad (3.4)$$

with the initial condition $S_t = s := x$. Then, we have an approximation formula $C = C_0 + \epsilon^{1/2} C_1 + O(\epsilon)$, where

$$\begin{aligned} C_0(t, x) &= E^{S_t} \left[e^{-r(T-t)} h(S_T) \right], \\ C_1(t, x) &= E^{S_t} \left[\int_t^T e^{-r(s-t)} g(s, S_s) ds \right], \\ g(t, x) &:= (A_1 - 2A_2)x^2 \frac{\partial^2 C_0}{\partial x^2} - A_2 x^3 \frac{\partial^3 C_0}{\partial x^3}. \end{aligned} \quad (3.5)$$

Moreover, (3.5) leads to

$$\frac{\partial C(t, x)}{\partial x} = E^{S_t} \left[e^{-r(T-t)} h(S_T) \pi_0^\Delta(T-t, x) \right] + \epsilon^{\frac{1}{2}} E^{S_t} \left[\int_t^T e^{-r(s-t)} g(s, S_s) \pi_1^\Delta(s-t, x) ds \right] + O(\epsilon), \quad (3.6)$$

where $\pi_0^\Delta(T-t, x) = W_{T-t}/\{x\bar{\sigma}(T-t)\}$ and $\pi_1^\Delta(s-t, x) = W_{s-t}/\{x\bar{\sigma}(s-t)\}$.

Proof: The infinitesimal generator of S_t is

$$rx \frac{\partial}{\partial x} + \frac{1}{2} \bar{\sigma}^2 x^2 \frac{\partial^2}{\partial x^2}. \quad (3.7)$$

Therefore, we have a probabilistic representation of the solution by the Feynman-Kac theorem. Since S_t follows the Black-Scholes model, we obtain an explicit formula for the leading term, $\partial C_0/\partial x$, by applying Theorem 1 and elementary calculation of the first variation process of S_t . (We choose a deterministic function $a(\tau)$ in Theorem 1 as $1/(T-t)$. According to Fouque *et al.* (2003), $C = C_0 + \epsilon^{1/2} C_1 + O(\epsilon)$ and terms C_0 and C_1 are smooth and bounded. Thus,

$$\frac{\partial}{\partial x} E^{S_t} \left[\int_t^T g(s, S_s) ds \right] = \int_t^T \frac{\partial}{\partial x} E^{S_t} [g(s, S_s)] ds.$$

To apply Theorem 1, we take $a(\tau) = 1/(s-t)$ i.e. $\int_t^s a(\tau) d\tau = 1$. Then we get (3.6). $\square$

Remark 2. By the integration-by-part formula (refer to Fournié (1999)), we have representations of Gamma and Vega in the following.

$$\frac{\partial^2 C(t, x)}{\partial x^2} = E^{S_t} \left[e^{-r(T-t)} (S_T - K)^+ \pi_0^\gamma(T-t, x) \right] + \epsilon^{\frac{1}{2}} E^{S_t} \left[\int_t^T e^{-r(s-t)} g(s, S_s) \pi_1^\gamma(s-t, x) ds \right] + O(\epsilon), \quad (3.8)$$

$$\frac{\partial C(t, x)}{\partial \bar{\sigma}} = E^{S_t} \left[e^{-r(T-t)} (S_T - K)^+ \pi_0^{\bar{\sigma}}(T-t, x) \right] + \epsilon^{\frac{1}{2}} E^{S_t} \left[\int_t^T e^{-r(s-t)} g(s, S_s) \pi_0^\sigma(s-t, x) ds \right] + O(\epsilon), \quad (3.9)$$

where

$$\begin{aligned} \pi_0^\gamma(T-t, x) &:= \frac{1}{x^2 \bar{\sigma}(T-t)} \left(\frac{(W_{T-t}^{(0)})^2}{\bar{\sigma}(T-t)} - W_{T-t}^{(0)} - \frac{1}{\bar{\sigma}} \right), \\ \pi_1^\gamma(s-t, x) &:= \frac{1}{x^2 \bar{\sigma}(s-t)} \left(\frac{(W_{s-t}^{(0)})^2}{\bar{\sigma}(s-t)} - W_{s-t}^{(0)} - \frac{1}{\bar{\sigma}} \right), \\ \pi_0^{\bar{\sigma}}(T-t, x) &:= \frac{(W_{T-t}^{(0)})^2}{\bar{\sigma}(T-t)} - W_{T-t}^{(0)} - \frac{1}{\bar{\sigma}}, \\ \pi_1^{\bar{\sigma}}(s-t, x) &:= \frac{(W_{s-t}^{(0)})^2}{\bar{\sigma}(s-t)} - W_{s-t}^{(0)} - \frac{1}{\bar{\sigma}}. \end{aligned}$$

Under this stochastic volatility model, a direct application of Fournié *et al.* (1999, 2001) leads a bad convergence rate in the numerical simulation of Malliavin Greeks. Since the model is fast-mean reverting i.e., $1/\epsilon$ is sufficiently large, the first variation process and an optimal weight of (3.1) should

have large variances. This implies that random variables in Malliavin formulas, such as $h(X_T)\pi^\Delta$ of (2.17), also have large variances. Therefore, the Malliavin Greeks have a low level convergence rate in numerical simulation. Moreover, the volatility function $f(v)$ which is estimated by market data may not guarantee the existence of the first variation process.

However, the above asymptotic approach does not contain the volatility function $f(v)$ and the large coefficient $1/\epsilon$. We only need constants $\bar{\sigma} = \sqrt{\langle f^2 \rangle}$ and $A_{1,2}$, not $f(v)$ nor $1/\epsilon$. This implies that the variance of a random process $e^{-r(T-t)}h(S_T)\pi_0^\Delta(T-t, x)$ in (3.6) is adequate for numerical simulation.

Next, we rewrite the formula (3.6) by replacing partial derivative terms $\partial^2 C_0/\partial x^2$ and $\partial^3 C_0/\partial x^3$ with alternative representations. In a European Call and a Put option cases, we already know explicit formulae for the Gamma and the Speed (3rd derivatives with respect to underlying). However, they are generally unknown in other cases. Therefore, we need to find a suitable representation with respect to high order partial derivatives such as Gamma and Speed which appear in the correction terms. By applying the Black-Scholes model, we have the following.

Theorem 3. *The first correction term $E^{S_t}[\int_t^T e^{-r(s-t)}g(s, S_s)\pi_1^\Delta(s-t, x)ds]$ of the formula (3.6) is reformulated by*

$$E^{S_t} \left[H(S_T) \int_t^T e^{-r(s-t)} \Gamma(T, t, x, s, S_s) ds \right], \quad (3.10)$$

where $H(x) = e^{-r(T-t)}(x - K)^+$ and

$$\begin{aligned} \Gamma(T, t, x, s, S_s) &:= \pi_1^\Delta(s-t, x) \left((A_1 + 2A_2)S_s^2 \pi_0^\gamma(T-s, S_s) + A_2 S_s^3 \pi_0^S(T-s, S_s) \right), \\ \pi_0^S(T-t, x) &:= \frac{(W_{T-t}^{(0)})^3}{x^2 \bar{\sigma}^2 (T-t)^2} - (W_{T-t}^{(0)})^2 \left(\frac{2}{x^3 \bar{\sigma}^2 (T-t)^2} + \frac{1}{x^2 \bar{\sigma}^2 (T-t)} \right) \\ &\quad + W_{T-t}^{(0)} \left(\frac{2}{x^3 \bar{\sigma} (T-t)} - \frac{2}{x^2 \bar{\sigma}^2 (T-t)} - \frac{1}{\bar{\sigma}} \right) + \frac{2}{x^3 \bar{\sigma}^2 (T-t)} + \frac{1}{x^2 \bar{\sigma}}. \end{aligned}$$

Proof: First, we show that $\partial^3 C_0/\partial x^3 = E^{S_t}[H(S_T)\pi_0^S(T-t, x)]$. Let $S_T^x = x + \int_t^T r S_s ds + \int_t^T \bar{\sigma} S_s dW_s^{(0)}$, which means that the process S_T starts from x at time t . By the integration by part formula for gamma, we have

$$\begin{aligned} \frac{\partial^3 C_0}{\partial x^3} &= \frac{\partial}{\partial x} \frac{\partial^2 C_0}{\partial x^2} = \frac{\partial}{\partial x} E^{S_t} \left[H(S_T^x) \pi_0^\gamma(T-t, x) \right] \\ &= E^Q \left[\frac{\partial H(S_T^x)}{\partial x} \pi_0^\gamma(T-t, x) + H(S_T^x) \frac{\partial \pi_0^\gamma}{\partial x}(T-t, x) | S_t = x \right] \\ &= E^Q \left[H'(S_T^x) \frac{\partial S_T^x}{\partial x} \pi_0^\gamma(T-t, x) | S_t = x \right] + E^Q \left[H(S_T^x) \frac{\partial \pi_0^\gamma}{\partial x}(T-t, x) | S_t = x \right]. \end{aligned}$$

Using the integration by part formula, we have

$$\begin{aligned} E^Q \left[H'(S_T^x) \frac{\partial S_T^x}{\partial x} \pi_0^\gamma(T-t, x) | S_t = x \right] &= E^Q \left[\int_t^T D_s H(S_T^x) \frac{\pi_0^\gamma(T-t, x)}{x \bar{\sigma} (T-t)} ds | S_t = x \right] \\ &= E^Q \left[H(S_T^x) \int_t^T \frac{\pi_0^\gamma(T-t, x)}{x \bar{\sigma} (T-t)} \delta W_s^{(0)} | S_t = x \right]. \end{aligned}$$

Figure 1: Error of call option price.

Figure 2: Malliavin Delta of leading term $\partial C_0 / \partial X_t$: Delta of Black-Sholes model at $X_t = K$.

Using the Wiener-Itô Chaos expansion to calculate the term $\int_t^T \{\pi_0^\gamma(T-t, x)\} / \{x\bar{\sigma}(T-t)\} \delta W_s^{(0)}$, we can check that

$$E^Q \left[H(S_T) \left(\int_t^T \frac{\pi_0^\gamma(T-t, x)}{x\bar{\sigma}(T-t)} \delta W_s^{(0)} + \frac{\partial \pi_0^\gamma}{\partial x}(T-t, x) \right) \middle| S_t = x \right] = E^{S_t} [H(S_T) \pi_0^S(T-t, x)].$$

However, the strong Markov property of S_t , $t \leq s \leq T$ leads to

$$\begin{aligned} & E^{S_t} [g(s, S_s) \pi_1^\Delta(s-t, x)] \\ &= E^Q \left[\left\{ (A_1 - 2A_2) S_s^2 \frac{\partial^2 C_0}{\partial x^2}(s, S_s) - A_2 S_s^3 \frac{\partial^3 C_0}{\partial x^3}(s, S_s) \right\} \pi_1^\Delta(s-t, x) \middle| S_t = x \right] \\ &= E^Q \left[(A_1 - 2A_2) S_s^2 E^Q [H(S_T) \pi_0^\gamma(T-s, S_s) | S_s] \pi_1^\Delta(s-t, x) \middle| S_t = x \right] \\ &\quad - E^Q \left[A_2 S_s^3 E^Q [H(S_T) \pi_0^S(T-s, S_s) | S_s] \pi_1^\Delta(s-t, x) \middle| S_t = x \right] \\ &= E^Q \left[E^Q \left[(A_1 - 2A_2) S_s^2 H(S_T) \pi_0^\gamma(T-s, S_s) \pi_1^\Delta(s-t, x) \middle| S_s \right] \middle| S_t = x \right] \\ &\quad - E^Q \left[E^Q \left[A_2 S_s^3 H(S_T) \pi_0^S(T-s, S_s) \pi_1^\Delta(s-t, x) \middle| S_s \right] \middle| S_t = x \right] \\ &= E^Q \left[(A_1 - 2A_2) S_s^2 H(S_T) \pi_0^\gamma(T-s, S_s) \pi_1^\Delta(s-t, x) \middle| S_t = x \right] \\ &\quad - E^Q \left[A_2 S_s^3 H(S_T) \pi_0^S(T-s, S_s) \pi_1^\Delta(s-t, x) \middle| S_t = x \right]. \end{aligned}$$

By Fubini's theorem, we get the result. $\square$

4. Numerical results

In this section, we present numerical performance of our method. We consider a European call option with the exercise value $K = 100$ and the expiry date $T - t = 1$. Under a risk-neutral measure, we assume that the underlying X_t satisfies (3.1). Let $f(v) = 0.2 * e^v$ and $\Lambda(v) = 0$. Set $r = 0.04$, $\epsilon = 0.01$, $m = 0$, $\nu = 0.4$, and $\rho = 0$. In this setting, we compare the Malliavin Delta of (3.6) at $X_t = K = 100$ to a finite difference Delta $\{u(x + \Delta x) - u(x - \Delta x)\} / (2\Delta x)$. We also compute an asymptotic delta which is given by Fouque *et al.* (2000, 2003, 2011) as the reference value since we do not know an exact formula of the Delta under the model (3.1). First, we simulate the price difference between an asymptotic option price and the Monte-Carlo simulation price. As in Figure 1, the asymptotic option price has $O(\sqrt{\epsilon})$ error, within 0.05. This implies that our test performs well for

Figure 3: Call option Delta, $X_t = K$. Figure 4: First correction: $\partial C_1/\partial X_t$, $X_t = K$.

comparing Delta of (3.6) to that of finite difference method. Figure 2 presents a Malliavin Delta of the Black-Scholes model. According to Figure 2, Malliavin Delta requires $1.2 \cdot 10^5$ times simulations for sufficient approximation. Figure 3 shows that the reference Delta under the stochastic volatility is close to 0.626. Unfortunately, a finite difference delta has a bad convergence performance. However, Malliavin approach needs $1.2 \cdot 10^5$ simulations to obtain 0.04% accuracy. Therefore, we can reduce the time cost by applying our method.

Figures 2 and 4 imply that the accuracy of the Malliavin Delta under the stochastic volatility model depends on the accuracy of leading term $\partial C_0/\partial X_t$. Figure 4 shows the accuracy of the first correction $\partial C_1/\partial X_t$. We notice that $\sqrt{\epsilon} \partial C_0/\partial X_t$ also requires $2 \cdot 10^4$ times simulations to reach a

Figure 5: Call option Gamma, $X_t = K$.

suitable accuracy. Therefore, we can obtain proper approximation, if Malliavin Greeks of the leading term $\partial C_0 / \partial X_t$ has less error. In this point of view, the study of leading term Greeks is also an issue in this field. It will be discussed in future work.

Finally, we compare the second derivative Gamma driven by Malliavin calculus to a finite difference Gamma. In Figure 5, we present a finite difference Gamma and Malliavin Gamma under the same setting. Here, we use the formula $\{C(x + \Delta x) + C(x - \Delta x) - 2C(x)\} / (\Delta x)^2$ for the finite difference Gamma. Likewise the finite difference Gamma in Figure 5, finite difference second derivatives are unstable in general. However, Malliavin Gamma has a similar convergence rate with respect to the Delta. Therefore, the Malliavin approach is efficient to compute high order derivatives.

5. Conclusion

In this paper, we studied an asymptotic computation method of Greeks under a stochastic volatility model. We utilized the Malliavin calculus and the singular perturbation method to derive a asymptotic formula of Greeks. Based on the Black-Scholes model, Greeks of stochastic volatility model are decomposed by two terms, the Greek of constant volatility model and a correction term. We also presented numerical simulations for the validation of the result. This simulation result shows that asymptotic Greeks has efficient performance rather than finite difference approximation. Especially, asymptotic Greeks provides better performance in the case of the Gamma simulation. This approach provides a stable computation method than the finite difference approximation for Greeks under the stochastic volatility model.

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