Inference for exponentiated Weibull distribution under constant stress partially accelerated life tests with multiple censored

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Abstract

Constant stress partially accelerated life tests are studied according to exponentiated Weibull distribution. Grounded on multiple censoring, the maximum likelihood estimators are determined in connection with unknown distribution parameters and accelerated factor. The confidence intervals of the unknown parameters and acceleration factor are constructed for large sample size. However, it is not possible to obtain the Bayes estimates in plain form, so we apply a Markov chain Monte Carlo method to deal with this issue, which permits us to create a credible interval of the associated parameters. Finally, based on constant stress partially accelerated life tests scheme with exponentiated Weibull distribution under multiple censoring, the illustrative example and the simulation results are used to investigate the maximum likelihood, and Bayesian estimates of the unknown parameters.

Keywords: constant stress partially accelerated life tests, exponentiated Weibull distribution, multiple censored scheme, maximum likelihood estimation, Bayesian estimation, Markov chain Monte Carlo

1. Introduction

Under reliability development and continuous quality of products make it difficult to grasp failure information under normal condition. A lifetime test under such conditions therefore becomes sluggish and futile over a long time span. An accelerated life test (ALT) is being used to quickly acquire information about the lifetime distribution of a product or materials. Accelerated life testing could be realized by submitting the test units to conditions that are more intense than what the normal ones are in imposing higher levels of pressure, temperature, vibration, voltage, load, and cycling rate. Research work on ALT began in the 1950's to advance a more operative testing procedure. Chernoff (1962) and Bessler *et al.* (1962) innovated and considered the concept of ALTs. In accelerated life testing, the experiment is to be initiated both at higher stresses than standard ones and sustained under the stated conditions or it may be under normal conditions. Hence, there are two types of accelerated life testing. The first is the ordinary accelerated life test (OALT), while the second one is the partially accelerated life testing (PALT). The chief supposition in OALT is that the mathematical model linking the lifetime of the unit to the stress must be recognized or can be anticipated; therefore, these models do not occur or are very difficult to assume in certain cases. PALT is practical to execute the life test. The chief supposition in PALT is that the mathematical pattern associating the mean lifetime and the

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stress are not known and cannot be suspected. PALT is applied for difficulties where it is a suitable test only at a particularized accelerated state, and then we extrapolated the data to a normal state. In such cases, PALT is the reasonable structure to implement and estimate the acceleration factor, which is the rate of the hazard level at the accelerated state to a normal use state ($\beta > 1$). There are two different types of PALTs. The first is a step-stress partially accelerated life testing (SS-PALT) that lets the test be modified from a normal state to accelerated state at a pre-defined time. The second is the constant-stress partially accelerated life testing (CS-PALT) where every item happens only in either normal condition or accelerated conditions.

The stress imposed in a PALT can be conceived in different methods. They comprise step stress, constant stress, and random stress. One approach to accelerate failures is constant stress in which each test item is only run at both a normal use state or accelerated state, i.e., each item is developed at a perpetual stress level until the test is completed. Nelson (1990) explained that constant stress testing has many benefits: First, it is simpler to conserve a constant stress level in most tests. Second, accelerated test patterns for constant stress are better to develop for some elements and products. Thirdly, data investigation for reliability estimation is well developed. The stress imposed in a PALT can be conceived in different methods. They comprise step stress, constant stress, and random stress. One approach to accelerated failures is constant stress in which each test item is run at both normal use state or accelerated state only, i.e., each item is developed at a perpetual stress level until the test is completed. Nelson (1990) explained that constant stress testing has many benefits: First, it is simpler to conserve a constant stress level in most tests. Second, accelerated test patterns for constant stress are better to develop some elements and products. Thirdly, data investigation for reliability estimation is well developed.

In life test experiments, sometimes the experiment could not get under control fully because items may break by mistake. Even though, Type I and Type II censoring schemes do not tolerate items to be eliminated from the test during the life testing duration. Progressive censoring schemes allow for items to be neglected only under control conditions. However, multiple censoring schemes are a suitable in this situation. Multiple censoring allows for items to be extracted from the test at any time during the life test duration. Multiple censoring may also; occur when the testing component fails for more than one reason. Hence Type I and II are exceptional cases of numerous censoring (Tobias and Trindade, 1995).

There is an aggregate of collected works on designing CS-PALT. Bai and Chung (1992) estimated the scale parameter and acceleration factor for exponential distribution under two kinds of PALT which are a step and constant stresses in case of Type I censored sample utilizing maximum likelihood method. Abdel-Ghani (1998) reflected that the estimating problems of Weibull distribution parameters and accelerated factor under CS-PALT. Ismail (2006) provided the optimum design of CS-PALT under Type II censoring supposing that the lifetime design stress has a Weibull distribution. Hassan (2007) estimated accelerated factor and the unknown parameters for generalized exponential distribution under CS-PALT using a maximum likelihood method. Ismail *et al.* (2011) used maximum likelihood methodology to estimate the accelerated factor and parameters for Pareto distribution with Type I censored data. Cheng and Wang (2012) used two maximum likelihood estimation methods, which accompany observed data and complete data likelihood function respectively, to estimate the accelerated factor and the unknown parameters for Burr XII distribution under CS-PALT. Zarrin *et al.* (2012) manipulated the maximum likelihood approach to estimate the accelerated factor and the parameters of Rayleigh's distribution. This work was done the CS-PALT under Type I censored data.

Kamal *et al.* (2013) manipulated the maximum likelihood approach to obtain the point and interval estimate for the accelerated factor and unknown parameters for inverted Weibull distribution under

CS-PALT in case type I censoring. Srivastava and Mittal (2013a, 2013b) used an optimal design of the CS-PALT plan under a failure constant and time constant for truncated logistic distribution based on type I censoring data. They determined the optimal sample proportion allocated to both normal and accelerated use conditions in the generalized asymptotic variance of maximum likelihood estimators (MLEs) for the accelerated factor and the unknown parameters. Jaheen *et al.* (2014) estimated the unknown parameters and accelerated factor for generalized exponential distribution under constant stress partially accelerated life testing with progressive Type II censored data using maximum likelihood and Bayesian approaches. Abushal and Soliman (2015) applied constant stress partially accelerated life testing with Type II progressively censored data to estimate unknown parameters and the accelerated factor for Pareto distribution using maximum likelihood and Bayesian methods. Hassan *et al.* (2015) used the maximum likelihood estimation to estimate the accelerated factor and the unknown parameters for inverted Weibull distribution under CS-PALT established on multiple censored data.

This paper is concerned with the estimation problem in the case of an exponentiated Weibull (EW) distribution under CS-PALT using multiple censored data. The rest of this paper constructed as follows. Section 2 introduces the EW distribution as the lifetime model, and the assumptions of the CS-PALT are presented. Section 3 displays the estimates of the unknown parameters and acceleration factor for the EW distribution with multiple censored samples. In addition, the confidence intervals of the unknown parameters are developed. The Bayesian estimators of the unknown parameters and acceleration factor for the EW distribution and Markov chain Monte Carlo (MCMC) technique are presented in Section 4. Section 5 presents a numerical example to illustrate all methods of inference developed in this article. Section 6 gives the simulation studies and the results. The conclusions are presented in Section 7.

2. The model and test procedure

This section presents the assumption pattern for product life test. The details of the test method of how to manipulate the experiment under multiple censored data using the EW model are illuminated.

2.1. The exponentiated Weibull distribution: as a failure time pattern

Mudholkar and Srivastava (1993) introduced an expansion distribution of the renowned Weibull distribution called EW distribution. The EW family includes distribution with monotone failure ratios aside from a broader class of monotone failure ratios. In practice, various lifetime data are of bathtub shape or upside-down bathtub shape failure ratios, and so the EW distribution as a failure pattern is more practical than monotone failure ratios and performs an important part to describe such data. The test procedure of CS-PALT based on multiple censored data assuming the life time item has EW distribution is described as

$$f_1(t_i) = \alpha \theta t_i^{\alpha-1} e^{-t_i^\alpha} (1 - e^{-t_i^\alpha})^{\theta-1}; \quad \alpha, \theta > 0, i = 1, 2, \dots, n_1 \quad (2.1)$$

and

$$F_1(t_i) = (1 - e^{-t_i^\alpha})^\theta, \quad (2.2)$$

where t_i is the i^{th} observed life time of test item at normal condition. Under accelerated condition, the probability density function (pdf) and cumulative distribution function of lifetime $X = \beta^{-1}T$, where β

is accelerated factor, $\beta > 1$, take the following forms:

$$f_2(x_j) = \alpha\beta\theta(\beta x_j)^{\alpha-1} e^{-(\beta x_j)^\alpha} (1 - e^{-(\beta x_j)^\alpha})^{\theta-1}; \quad \alpha, \theta > 0, \beta > 1, j = 1, 2, 3, \dots, n_2 \quad (2.3)$$

and

$$F_2(x_j) = (1 - e^{-(\beta x_j)^\alpha})^\theta, \quad (2.4)$$

where x_j is the j^{th} observed life time of test item at accelerated condition.

2.2. Basic assumptions

The basic assumptions are:

- (i) The lifetimes of items T_i , $i = 1, 2, \dots, n_1$ devoted to normal condition are independent and identically distributed random variables with pdf (2.1).
- (ii) The lifetimes of items X_j , $j = 1, 2, \dots, n_2$ devoted to accelerated condition are independent and identically distributed random variables with pdf (2.3).
- (iii) The lifetimes T_i and X_j are mutually independent.

3. Maximum likelihood estimators

Suppose that the observed values of the total lifetime T at normal condition are $t_{(1)} < \dots < t_{(n_1)}$, and the observed values of the total lifetime X at accelerated condition are $x_{(1)} < \dots < x_{(n_2)}$. The likelihood function for the unknown parameters $\phi = (\alpha, \theta, \beta)$ of the EW distribution with multiple censored data is given as

$$L(\phi|x) = \Pi_{i=1}^n [f_1(t_i)]^{\gamma_{i,(1,f)}} [1 - F_1(t_i)]^{\gamma_{i,(1,c)}} [f_2(x_i)]^{\gamma_{i,(2,f)}} [1 - F_2(x_i)]^{\gamma_{i,(2,c)}}, \quad (3.1)$$

where $t_i = t_{(i)}$, $x_i = x_{(i)}$ for simplicity of notation, and $\gamma_{i,(1,f)}$, $\gamma_{i,(1,c)}$, $\gamma_{i,(2,f)}$, $\gamma_{i,(2,c)}$ be the indicator functions, $i = 1, 2, \dots, n$, such that,

$$\begin{aligned} \gamma_{i,(1,f)} &= \begin{cases} 1, & \text{the unit fails at normal condition,} \\ 0, & \text{otherwise,} \end{cases} \\ \gamma_{i,(1,c)} &= \begin{cases} 1, & \text{the unit censoring at normal condition,} \\ 0, & \text{otherwise,} \end{cases} \\ \gamma_{i,(2,f)} &= \begin{cases} 1, & \text{the unit fails at stress condition,} \\ 0, & \text{otherwise,} \end{cases} \\ \gamma_{i,(2,c)} &= \begin{cases} 1, & \text{the unit censoring at stress condition,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

Moreover, the equations are obtained as:

$$\begin{aligned} \sum_{i=1}^n \gamma_{i,(1,f)} &= n_{1f}, \quad \sum_{i=1}^n \gamma_{i,(1,c)} = n_{1c}, \quad \sum_{i=1}^n \gamma_{i,(2,f)} = n_{2f}, \quad \sum_{i=1}^n \gamma_{i,(2,c)} = n_{2c}, \\ n_f &= n_{1f} + n_{2f}, \quad n_1 = n_{1f} + n_{1c}, \quad n_2 = n_{2f} + n_{2c}, \quad n = n_1 + n_2. \end{aligned}$$

Substituting the probability density and cumulative distribution functions (2.1), (2.2), (2.3), and (2.4) in likelihood function (3.1), then:

$$L(\phi|x) = \prod_{i=1}^n \left[\alpha \theta t_i^{\alpha-1} e^{-t_i^\alpha} \left(1 - e^{-t_i^\alpha}\right)^{\theta-1} \right]^{\gamma_{i(1,f)}} \left[1 - \left(1 - e^{-t_i^\alpha}\right)^\theta \right]^{\gamma_{i(1,c)}} \\ \times \left[\alpha \beta \theta (\beta x_i)^{\alpha-1} e^{-(\beta x_i)^\alpha} \left(1 - e^{-(\beta x_i)^\alpha}\right)^{\theta-1} \right]^{\gamma_{i(2,f)}} \left[1 - \left(1 - e^{-(\beta x_i)^\alpha}\right)^\theta \right]^{\gamma_{i(2,c)}}. \quad (3.2)$$

Hence, it is usually easier to maximize the natural logarithm of the likelihood function instead of the likelihood function itself, the logarithm of likelihood function is considered as

$$\ln L = n_f \ln \alpha + n_f \ln \theta + n_{2f} (\ln \beta - \beta^\alpha) - \sum_{i=1}^n \gamma_{i(1,f)} t_i^\alpha - \sum_{i=1}^n \gamma_{i(2,f)} x_i^\alpha \\ + (\alpha - 1) \left[\sum_{i=1}^n \gamma_{i(1,f)} \ln t_i + \sum_{i=1}^n \gamma_{i(2,f)} \ln(\beta x_i) \right] + \sum_{i=1}^n \gamma_{i(1,c)} \ln \left[1 - u_{1i}^\theta \right] \\ + (\theta - 1) \left[\sum_{i=1}^n \gamma_{i(1,f)} \ln(u_{1i}) + \sum_{i=1}^n \gamma_{i(2,f)} \ln(u_{2i}) \right] + \sum_{i=1}^n \gamma_{i(2,c)} \ln \left[1 - u_{2i}^\theta \right], \quad (3.3)$$

where $u_{1i} = (1 - e^{-t_i^\alpha})$ and $u_{2i} = (1 - e^{-(\beta x_i)^\alpha})$.

Maximum likelihood estimators of $\phi = (\alpha, \theta, \beta)$ are a solution to the system of equations achieved by addressing the first partial derivatives of the total log-likelihood to be zero concerning α , θ , and β respectively. Thus, the system of equations is considered as:

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n_f}{\alpha} - n_{2f} \beta^\alpha \ln \beta - \sum_{i=1}^n \gamma_{i(1,f)} t_i^\alpha \ln t_i - \sum_{i=1}^n \gamma_{i(2,f)} x_i^\alpha \ln x_i + \sum_{i=1}^n \gamma_{i(1,f)} \ln t_i + \sum_{i=1}^n \gamma_{i(2,f)} \ln(\beta x_i) \\ + (\theta - 1) \left[\sum_{i=1}^n \gamma_{i(1,f)} \varphi_{1i} + \sum_{i=1}^n \gamma_{i(2,f)} \varphi_{2i} \right] - \theta \left[\sum_{i=1}^n \gamma_{i(1,c)} \varphi_{3i} + \sum_{i=1}^n \gamma_{i(2,c)} \varphi_{4i} \right], \quad (3.4)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n_f}{\theta} + \sum_{i=1}^n \gamma_{i(1,f)} \ln(u_{1i}) + \sum_{i=1}^n \gamma_{i(2,f)} \ln(u_{2i}) - \sum_{i=1}^n \gamma_{i(1,f)} \psi_{1i} - \sum_{i=1}^n \gamma_{i(2,c)} \psi_{2i}, \quad (3.5)$$

$$\frac{\partial \ln L}{\partial \beta} = n_{2f} \beta^{-1} (1 - \alpha \beta^\alpha) + (\alpha - 1) \sum_{i=1}^n \gamma_{i(2,f)} x_i (\beta x_i)^{-1} + \alpha (\theta - 1) \sum_{i=1}^n \gamma_{i(2,c)} \xi_{1i} - \alpha \theta \sum_{i=1}^n \gamma_{i(2,c)} \xi_{2i}, \quad (3.6)$$

where

$$\begin{pmatrix} \varphi_{1i} \\ \varphi_{2i} \\ \varphi_{3i} \\ \varphi_{4i} \\ \psi_{1i} \\ \psi_{2i} \\ \xi_{1i} \\ \xi_{2i} \end{pmatrix} = \begin{pmatrix} e^{-t_i^\alpha} t_i^\alpha \ln t_i u_{1i}^{-1} \\ e^{-(\beta x_i)^\alpha} (\beta x_i)^\alpha \ln(\beta x_i) u_{2i}^{-1} \\ \varphi_{1i} u_{1i}^\theta \left[1 - u_{1i}^\theta \right]^{-1} \\ \varphi_{2i} u_{2i}^\theta \left[1 - u_{2i}^\theta \right]^{-1} \\ u_{1i}^\theta \ln(u_{1i}) \left[1 - u_{1i}^\theta \right]^{-1} \\ u_{2i}^\theta \ln(u_{2i}) \left[1 - u_{2i}^\theta \right]^{-1} \\ x_i (\beta x_i)^{\alpha-1} e^{-(\beta x_i)^\alpha} u_{2i}^{-1} \\ x_i (\beta x_i)^{\alpha-1} e^{-(\beta x_i)^\alpha} u_{2i}^{\theta-1} \left[1 - u_{2i}^\theta \right]^{-1} \end{pmatrix}.$$

Since, the closed form solutions to the nonlinear Equations (3.4), (3.5), and (3.6) are difficult to reach. The numerical method is then used to solve these simultaneous equations for obtaining $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\beta}$,

the second partial derivative of the matrix likelihood function are give as:

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \alpha^2} &= -\frac{n_f}{\alpha^2} - n_{2f} \beta^\alpha (\ln \beta)^2 - \sum_{i=1}^n \gamma_{i,(1,f)} t_i^\alpha (\ln t_i)^2 - \sum_{i=1}^n \gamma_{i,(2,f)} x_i^\alpha (\ln x_i)^2 \\
&\quad + (\theta - 1) \left[\sum_{i=1}^n \gamma_{i,(1,f)} \varphi_{1i} \varpi_{1i} + \sum_{i=1}^n \gamma_{i,(2,f)} \varphi_{2i} \varpi_{2i} \right] \\
&\quad - \theta \left\{ \sum_{i=1}^n \gamma_{i,(1,c)} \varphi_{3i} [\varpi_{1i} + \theta(\varphi_{1i} + \varphi_{3i})] + \sum_{i=1}^n \gamma_{i,(2,c)} \varphi_{4i} [\varpi_{2i} + \theta(\varphi_{2i} + \varphi_{4i})] \right\}, \\
\frac{\partial^2 \ln L}{\partial \theta^2} &= -\frac{n_f}{\alpha^2} - \sum_{i=1}^n \gamma_{i,(1,c)} \psi_{1i} [\ln(u_{1i}) + \psi_{1i}] - \sum_{i=1}^n \gamma_{i,(2,c)} \psi_{2i} [\ln(u_{2i}) + u_{2i}^\theta \psi_{2i}], \\
\frac{\partial^2 \ln L}{\partial \beta^2} &= -n_{2f} \beta^{-2} [1 + \alpha(\alpha - 1) \beta^\alpha] - (\alpha - 1) \sum_{i=1}^n \gamma_{i,(2,f)} x_i^2 (\beta x_i)^{-2} \\
&\quad + \alpha(\theta - 1) \sum_{i=1}^n \gamma_{i,(2,f)} \xi_{1i} \varpi_{3i} - \alpha \theta \sum_{i=1}^n \gamma_{i,(2,c)} \xi_{2i} [\varpi_{3i} + \theta \alpha (\xi_{1i} + \xi_{2i})], \\
\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} &= \sum_{i=1}^n \gamma_{i,(1,f)} \varphi_{1i} + \sum_{i=1}^n \gamma_{i,(2,f)} \varphi_{2i} - \sum_{i=1}^n \gamma_{i,(1,c)} \varphi_{3i} [1 + \theta \ln(u_{1i}) + \theta \psi_{1i}] \\
&\quad - \sum_{i=1}^n \gamma_{i,(2,c)} \varphi_{4i} [1 + \theta \ln(u_{2i}) + \theta \psi_{2i}], \\
\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} &= -n_{2f} \beta^\alpha [\ln \beta + \beta^{-1}] + \sum_{i=1}^n \gamma_{i,(2,f)} x_i (\beta x_i)^{-1} + (\theta - 1) \sum_{i=1}^n \gamma_{i,(2,f)} \xi_{1i} \ln(\beta x_i) \varpi_{4i} \\
&\quad - \theta \sum_{i=1}^n \gamma_{i,(2,c)} \xi_{2i} \ln(\beta x_i) [\varpi_{4i} + \theta \alpha x_i^{-1} (\beta x_i) (\xi_{1i} + \xi_{2i})], \\
\frac{\partial^2 \ln L}{\partial \theta \partial \beta} &= \alpha \sum_{i=1}^n \gamma_{i,(2,f)} \xi_{1i} - \alpha \sum_{i=1}^n \gamma_{i,(2,c)} \xi_{2i} [\theta \ln(u_{2i}) + 1 + \psi_{2i}],
\end{aligned}$$

where

$$\begin{pmatrix} \varpi_{1i} \\ \varpi_{2i} \\ \varpi_{3i} \\ \varpi_{4i} \end{pmatrix} = \begin{pmatrix} \ln t_i - t_i^\alpha \ln t_i - \varphi_{1i} \\ \ln(\beta x_i) - (\beta x_i)^\alpha \ln(\beta x_i) - \varphi_{2i} \\ (\alpha - 1) x_i (\beta x_i)^{-1} - \alpha x_i (\beta x_i)^{\alpha-1} - \alpha \xi_{1i} \\ [\ln(\beta x_i)]^{-1} - \alpha (\beta x_i)^\alpha + \alpha - \alpha x_i^{-1} (\beta x_i) \xi_{1i} \end{pmatrix}.$$

That is, the interval estimation of the model parameters that occur in 3×3 observed information matrix $I(\phi) = [U_{r,s}]$ for $(r, s = \alpha, \theta \text{ and } \beta)$.

Frequently, following standard regularity conditions, the multivariate normal $N_3[0, I^{-1}(\hat{\phi})]$ distribution is applied to form approximate confidence intervals for the parameters. The approximate $100(1 - \Upsilon)\%$ two-sided confidence intervals for α , θ , and β are then given, respectively, by:

$$\hat{\alpha} \pm Z_{\Upsilon/2} \sqrt{\text{var}(\hat{\alpha})}, \quad \hat{\theta} \pm Z_{\Upsilon/2} \sqrt{\text{var}(\hat{\theta})}, \quad \hat{\beta} \pm Z_{\Upsilon/2} \sqrt{\text{var}(\hat{\beta})}.$$

Here, $Z_{\Upsilon/2}$ is the upper $(\Upsilon/2)^{th}$ percentile of the standard normal distribution and $\text{var}(\cdot)$'s indicates the diagonal elements of $I^{-1}(\hat{\phi})$ corresponding to the model parameters.

4. Bayesian estimation

In this section, the Bayesian estimator using a squared error loss function is considered. Following the assumption of gamma distribution to be a prior of two unknown parameters of the EW distribution will be concern based on multiple censored scheme. The Bayesian estimator under the assumption

that the random variables α and θ are independently distributed gamma prior distribution (with known shape and scale parameters ν , b , c , and d) considered as

$$g(\alpha) \propto \alpha^{\nu-1} e^{-\alpha b}, \quad \nu, b > 0 \quad \text{and} \quad g(\theta) \propto \theta^{c-1} e^{-\theta d}, \quad c, d > 0.$$

While the random variable β distributed as the following non-informative (NIP) type of prior

$$g(\beta) \propto \beta^{-1}, \quad \beta > 1.$$

Then, the joint prior density of unknown parameters $\phi = \alpha, \theta$, and β can be written as

$$\pi(\phi) \propto \beta^{-1} \alpha^{\nu-1} \theta^{c-1} e^{-(ab+\theta d)}, \quad \alpha, \theta > 0, \beta > 1. \quad (4.1)$$

Combining (3.2) and (4.1) to get the posterior density of ϕ as follows

$$\pi^*(\phi|x) = \frac{L(\phi|x)\pi(\phi)}{\int_1^\infty \int_0^\infty \int_0^\infty L(\phi|x)\pi(\phi) d\alpha d\theta d\beta}, \quad (4.2)$$

thus, the Bayesian estimators of function of the parameters, say $\phi = (\alpha, \theta, \beta)$ based on multiple censored scheme under squared error loss function; denoted by $\tilde{u}_{(\text{BESL})}(\phi)$ can be determined by the following equations

$$\tilde{\phi}_{(\text{BESL})} = E(\phi|x) = \frac{\int_1^\infty \int_0^\infty \int_0^\infty \phi L(\phi|x)\pi(\phi) d\alpha d\theta d\beta}{\int_1^\infty \int_0^\infty \int_0^\infty L(\phi|x)\pi(\phi) d\alpha d\theta d\beta}. \quad (4.3)$$

The ratio of four integrals provided by Equation (4.3) and the three integrals are not achieved in a closed form. It is often instructive to apply the MCMC method to generate samples from the posterior distributions and then calculate the Bayes estimates of the particular parameters.

This section focuses on the Bayesian approach to estimate the unknown parameters of the EW model under multiple censored scheme considering the Gibbs sampler method for approximation of integrals. Often integrals occurring in the Bayesian estimation never admit a closed form, as a result, the analytical method is not attractable and numerical integration is needed. Gibbs sampler give an alternative method to approximate these integrals, this procedure intends to obtain a Markov chain that has a purpose posterior as a limiting distribution, and then the simulation samples or chain can be utilized to estimate any desired feature. The simulation Markov chain (after some burn-in) will have achievements that are observed as simulated from the posterior distribution rather than the straight computation of the posterior. We then utilized the generated samples to estimate the parameters of interest and function.

The MCMC scheme is a good method for parameter estimation. Several schemes of MCMC are available, one significant sub-class of MCMC methods is Gibbs sampling and Metropolis within-Gibbs samplers. MCMC has an advantage over the maximum likelihood method by structuring the probability intervals based on the empirical posterior distribution, we get an appropriate interval estimate of the parameters. The MCMC samples utilized to summarize the posterior uncertainty about the parameters $\phi = (\alpha, \theta, \beta)$, within the joint posterior density function of α, θ and β can be addressed as

$$\begin{aligned} \pi^*(\phi|x) \propto & \alpha^{n_f+\nu-1} \theta^{n_f+c-1} \beta^{n_{2f}\alpha-1} e^{-[ab+\theta d+\sum_{i=1}^n \gamma_{i,(1,f)} t_i^\alpha + \sum_{i=1}^n \gamma_{i,(2,f)} (\beta x_i)^\alpha]} \left[\prod_{i=1}^n t_i^{(\alpha-1)(\gamma_{i,(1,f)})} \right] \\ & \times \left[\prod_{i=1}^n x_i^{(\alpha-1)(\gamma_{i,(2,f)})} \right] \left[\prod_{i=1}^n u_{1i}^{(\theta-1)(\gamma_{i,(1,f)})} \right] \left[\prod_{i=1}^n (1-u_{1i}^\theta)^{\gamma_{i,(1,c)}} \right] \left[\prod_{i=1}^n u_{2i}^{(\theta-1)(\gamma_{i,(2,f)})} \right] \\ & \times \left[\prod_{i=1}^n (1-u_{2i}^\theta)^{\gamma_{i,(2,c)}} \right], \quad \alpha, \lambda > 0, \beta > 1. \end{aligned} \quad (4.4)$$

The conditional posterior densities of α , θ and β can be computed by:

$$\begin{aligned}\pi_1^*(\alpha|x) &\propto \alpha^{n_f+v-1} \beta^{n_{2f}\alpha-1} e^{-[ab+\sum_{i=1}^n \gamma_{i,(1,f)} t_i^\alpha + \sum_{i=1}^n \gamma_{i,(2,f)} (\beta x_i)^\alpha]} \left[\prod_{i=1}^n t_i^{(\alpha-1)(\gamma_{i,(1,f)})} \right] \left[\prod_{i=1}^n x_i^{(\alpha-1)(\gamma_{i,(2,f)})} \right] \\ &\quad \times \left[\prod_{i=1}^n u_{1i}^{(\theta-1)(\gamma_{i,(1,f)})} \right] \left[\prod_{i=1}^n u_{2i}^{(\theta-1)(\gamma_{i,(2,f)})} \right] \left[\prod_{i=1}^n (1-u_{1i})^{\gamma_{i,(1,c)}} \right] \left[\prod_{i=1}^n (1-u_{2i})^{\gamma_{i,(2,c)}} \right], \\ \pi_2^*(\theta|x) &\propto \theta^{n_f+c-1} e^{-\theta d} \left[\prod_{i=1}^n u_{1i}^{(\theta-1)(\gamma_{i,(1,f)})} \right] \left[\prod_{i=1}^n u_{2i}^{(\theta-1)(\gamma_{i,(2,f)})} \right] \left[\prod_{i=1}^n (1-u_{1i})^{\gamma_{i,(1,c)}} \right] \left[\prod_{i=1}^n (1-u_{2i})^{\gamma_{i,(2,c)}} \right], \\ \pi_3^*(\beta|x) &\propto \beta^{n_{2f}\alpha-1} e^{-\sum_{i=1}^n \gamma_{i,(2,f)} (\beta x_i)^\alpha} \left[\prod_{i=1}^n u_{2i}^{(\theta-1)(\gamma_{i,(2,f)})} \right] \left[\prod_{i=1}^n (1-u_{2i})^{\gamma_{i,(2,c)}} \right].\end{aligned}$$

Consequently, to generate from these distributions, the Metropolis-Hastings method (Metropolis *et al.*, 1953) and Robert and Casella (2004), are utilized for details concerning the implementation of Metropolis-Hastings algorithm.

The procedure of Gibbs sampling can be described as:

Step 1: Start with an $(\alpha^{(0)} = \alpha, \theta^{(0)} = \theta)$ and $(\beta^{(0)} = \beta)$ and set $I = 1$.

Step 2: Generate α^I from $\pi_1^*(\alpha|x)$.

Step 3: Generate θ^I from $\pi_2^*(\theta|x)$.

Step 4: Generate β^I from $\pi_3^*(\beta|x)$.

Step 5: Compute α^I, θ^I and β^I .

Step 6: Set $I = I + 1$.

Step 7: Repeat Step 2–5 N times.

Step 8: Obtain the Bayes MCMC point estimate of ϕ_q ($\phi_1 = \alpha, \phi_2 = \theta, \phi_3 = \beta$), $q = 1, 2, 3$ as

$$E(\phi_q|\text{data}) \propto \frac{\left(\sum_{i=M+1}^N \phi_q^{(i)} \right)}{N - M},$$

where, M is the burn-in period (that is, some iterations before the stationary distribution perform) and the posterior variance of ϕ becomes

$$\widehat{V}(\phi_q|\text{data}) \propto \frac{\sum_{i=M+1}^N \left(\phi_q^{(i)} - E(\phi_q|\text{data}) \right)^2}{N - M}.$$

Step 9: To compute the credible intervals of ϕ_q , we use the quantiles of the sample as the endpoints of the interval. Order $\phi_q^{(M+1)}, \phi_q^{(M+2)}, \dots, \phi_q^{(N)}$ as $\phi_{q(1)}, \phi_{q(2)}, \dots, \phi_{q(N-M)}$. Consider the $100(1 - 2\Upsilon)\%$ symmetric credible interval as $[\phi_{q[\Upsilon(N-M)]}, \phi_{q[(1-\Upsilon)(N-M)}]$.

5. Illustrative example

Illustrative maximum likelihood estimation for the EW distribution based on another data set was simulated from distribution in CS-PALT scheme with $\alpha = 1.5, \theta = 3.5, \beta = 1.25$ the data are represented as follows.

- Under normal condition:

Figure 1: Profile log-likelihood function of θ .

0.296, 0.448, 0.449, 0.697, 0.791, 0.852, 0.896*, 0.907, 0.908*, 0.915, 0.918*, 0.922*, 0.942, 0.946*, 0.955, 0.984*, 0.988, 1.101, 1.116, 1.156, 1.222, 1.258, 1.327, 1.329*, 1.363, 1.373, 1.379, 1.412, 1.422*, 1.459*, 1.463, 1.480, 1.492, 1.542, 1.562, 1.566, 1.584, 1.588, 1.604, 1.717, 1.746*, 1.759*, 1.780, 1.829, 1.848, 1.849, 1.865*, 1.929, 1.938, 2.025*, 2.037*, 2.041, 2.084, 2.092, 2.164, 2.214, 2.668, 2.782*, 3.196, 3.658.

- Under stress condition:

0.249, 0.465, 0.608*, 0.638, 0.644*, 0.664, 0.666, 0.673, 0.688, 0.695*, 0.715, 0.734*, 0.737*, 0.770, 0.778, 0.786, 0.794, 0.813, 0.834, 0.853, 0.865, 0.897, 0.904, 0.921*, 0.927*, 0.937, 0.938*, 1.043*, 1.045, 1.063, 1.075, 1.075, 1.092, 1.097, 1.098, 1.107*, 1.114, 1.141, 1.148, 1.152, 1.167, 1.184, 1.198, 1.20*, 1.206, 1.21, 1.212, 1.225, 1.234, 1.238*, 1.240, 1.249, 1.266, 1.269, 1.275, 1.295*, 1.301, 1.305, 1.306, 1.323*, 1.346, 1.360*, 1.381, 1.401, 1.402, 1.442, 1.451*, 1.459, 1.469*, 1.471, 1.473, 1.493, 1.553*, 1.593*, 1.596, 1.641*, 1.660, 1.660, 1.672, 1.678, 1.681, 1.690, 1.701*, 1.711, 1.763*, 1.831, 1.895*, 1.985, 2.088, 2.437.

There are 150 samples with censoring level (CL) = 0.20. The π value, which is the proportion of the sample under normal condition is 0.40. The sample of failure and censoring in two groups based on CS-PALT, respectively, are 45 failures in group 1, 15 censoring in group 1, 67 failures in group 2 and 23 censoring in group 2. The symbol “*” denotes censored values. Before computing the MLEs, we plot the profile log-likelihood function in Figure 1. It is noted from Figure 1 that the profile log-likelihood function is unimodal. Then, the maximum likelihood estimates are $\hat{\alpha} = 1.257$, $\hat{\theta} = 3.672$, and $\hat{\beta} = 1.291$, and the corresponding 95% confidence intervals are (1.094, 1.271), (2.887, 4.457), and (1.093, 1.489). To compute the Bayes estimates of α , θ , and β , we assume the gamma prior distribution for (α, θ) and non-informative prior distribution for β . Based on MCMC samples of size 10,000, the Bayes estimates under SE loss function are $\tilde{\theta} = 1.331$, $\tilde{\theta} = 4.368$, and $\tilde{\beta} = 1.708$ and the corresponding credible intervals are (1.271, 1.731), (0.604, 5.104), and (1.700, 1.711), respectively.

6. Simulation study

In any estimation problems, it is required to study the properties of the derived estimators. The derived expressions for the estimators are too complicated to study analytically. Consequently, a

numerical study will be established and treat the sampling distribution of the estimators separately. A numerical study is achieved to compare the different estimates discussed in the previous section. The performances of the different estimates are compared regarding their mean squared error (MSE), relative bias (RB), relative error (RE), standard deviation (SD) and standard error (SE). The numerical procedures are:

Step 1: Divide the total sample size n in two sub samples $n_1 = n\pi$ and $n_2 = n(1 - \pi)$ where π is the sample proportion in normal condition.

Step 2: Generate n_1 random samples as normal condition samples $t_{1,1} < \dots < t_{n_1,1}$ from EW distribution which has cdf as $F_1(t) = (1 - e^{-t^\alpha})^\theta$; $\alpha, \theta > 0$. In addition; generate n_2 random samples as a stress condition samples $x_{1,2} < \dots < x_{n_2,2}$ from EW distribution which has cdf as $F_2(x) = (1 - e^{-(\beta x)^\alpha})^\theta$ with accelerated factor, where $\alpha, \theta > 0$, and $\beta > 1$.

Step 3: Select $n_{1f} = n_1 \times (1 - \text{CL})$, and $n_{2f} = n_2 \times (1 - \text{CL})$ to be the number of failure samples in the normal condition and in the stress condition, respectively, where CL is the censoring level. For example, $\text{CL} = 0.4$ indicates 60% failures and 40% censored data.

Step 4: Let

$$k_{i,1} = \begin{cases} 1, & i = 1, \dots, n_{1f}, \\ u_{i,1}, & i = n_{1f} + 1, \dots, n_1, \end{cases}$$

$$k_{j,2} = \begin{cases} 1, & j = 1, \dots, n_{2f}, \\ u_{j,1}, & j = n_{2f} + 1, \dots, n_2, \end{cases}$$

where, $u_{i,1}$ and $u_{j,2}$ are drawn from uniform distribution $U(0, 1)$.

Step 5: Set $y_{i,1} = k_{i,1} \times t_{i,1}$, $i = 1, 2, \dots, n_1$ and $y_{j,2} = k_{j,2} \times x_{j,2}$, $j = 1, 2, \dots, n_2$. Then, $y = [y_{i,1}, y_{j,2}; i = 1, 2, \dots, n_1; j = 1, 2, \dots, n_2]$ is generated as a multiple censored data for the EW distribution.

Step 6: 10,000 random samples of sizes 50, 100, 150, 300, and 500 are generated from EW distribution. The parameters values are chosen as case I $\equiv (\alpha = 2, \theta = 4, \beta = 1.2)$, case II $\equiv (\alpha = 1.5, \theta = 4, \beta = 1.2)$ and case III $\equiv (\alpha = 2, \theta = 4, \beta = 1.5)$. The CL is selected as 0.2, 0.3, and 0.5.

Step 7: For each sample and for the selected sets of parameters, the population parameters and the accelerated factor are estimated in CS-PALT under multiple censored samples. An iterative procedure is employed for solving the nonlinear Equations (3.4), (3.5), and (3.6) to obtain the estimates of α, θ , and β .

Step 8: The MSE, RB, RE, and SE of the estimators for acceleration factor and two shape parameters for all sample sizes and for three sets of parameters are classified.

Step 9: The confidence limit with confidence level and coverage probability (CP) of the acceleration factor and the two shape parameters are constructed at $\Upsilon = 0.95$ and $\Upsilon = 0.99$.

Tables 1–5 report the simulation results. Tables 1–3 represent the MSE, RB, RE, and SE for different cases of parameters and the approximated confidence intervals at 95% and 99% for the unknown parameters. In addition; the Bayes estimate and credible intervals for different cases of

Table 1: Results of simulation study of MSE, RB, RE, SE, and confidence interval (CI) of different values of parameters case $I \equiv (\alpha = 2, \theta = 4, \beta = 1.2)$ for different samples size

n	ϕ	Properties				95%				99%					
		MSE	RB	RE	SE	Lower	Upper	Length	CP	Lower	Upper	Length	CP		
CL = 0.2	50	α	0.185	0.215	0.215	3.377*	1.524	1.617	0.094	0.948	1.509	1.632	0.123	0.962	
		θ	0.438	0.047	0.165	0.090	2.946	5.432	2.486	0.956	2.553	5.826	3.273	0.988	
		β	0.015	0.089	0.103	8.858*	1.184	1.429	0.246	0.951	1.145	1.468	0.323	0.988	
	100	α	0.182	0.213	0.213	1.557*	1.543	1.604	0.061	0.954	1.534	1.614	0.080	0.967	
		θ	0.197	0.034	0.111	0.042	3.307	4.963	1.655	0.945	3.045	5.224	2.179	0.985	
		β	0.012	0.083	0.092	4.585*	1.210	1.390	0.180	0.951	1.182	1.418	0.237	0.991	
	150	α	0.180	0.212	0.212	6.396*	1.561	1.591	0.031	0.967	1.556	1.596	0.040	0.977	
		θ	0.139	0.029	0.093	0.029	3.426	4.810	1.384	0.954	3.207	5.029	1.822	0.987	
		β	0.011	0.082	0.089	3.112*	1.223	1.373	0.149	0.952	1.200	1.396	0.197	0.988	
	300	α	0.179	0.211	0.212	0.245*	1.569	1.585	0.017	0.974	1.566	1.588	0.022	0.984	
		θ	0.069	0.026	0.066	0.014	3.633	4.577	0.944	0.950	3.484	4.726	1.243	0.990	
		β	0.010	0.082	0.085	1.527*	1.246	1.350	0.104	0.950	1.230	1.366	0.136	0.995	
	500	α	0.179	0.211	0.211	0.007*	1.575	1.580	5.849*	0.961	1.574	1.581	7.699*	0.968	
		θ	0.046	0.026	0.054	8.397*	3.735	4.471	0.736	0.949	3.619	4.587	0.969	0.990	
		β	9.821*	0.081	0.083	9.345*	1.256	1.338	0.082	0.947	1.243	1.351	0.108	0.989	
	CL = 0.3	50	α	0.249	0.246	0.249	0.011	1.350	1.666	0.315	0.940	1.300	1.716	0.415	0.985
			θ	0.412	0.029	0.161	0.089	2.876	5.353	2.477	0.959	2.485	5.745	3.261	0.987
			β	6.270*	0.029	0.066	0.010	1.096	1.374	0.278	0.943	1.052	1.418	0.366	0.988
		100	α	0.180	8.932*	0.106	0.042	3.136	4.793	1.657	0.955	2.874	5.055	2.181	0.986
			θ	0.180	8.932*	0.106	0.042	3.136	4.793	1.657	0.955	2.874	5.055	2.181	0.986
			β	3.414*	0.026	0.049	4.921*	1.135	1.328	0.193	0.946	1.105	1.358	0.254	0.992
		150	α	0.234	0.241	0.242	4.448*	1.412	1.626	0.214	0.954	1.378	1.659	0.281	0.988
			θ	0.112	0.014	0.084	0.027	3.298	4.593	1.295	0.958	3.093	4.797	1.705	0.992
			β	2.398*	0.022	0.041	3.389*	1.145	1.307	0.163	0.947	1.119	1.333	0.214	0.992
300		α	0.226	0.237	0.238	2.338*	1.447	1.606	0.159	0.959	1.422	1.631	0.209	0.992	
		θ	0.061	0.020	0.062	0.014	3.461	4.380	0.919	0.945	3.316	4.525	1.210	0.986	
		β	1.535*	0.022	0.033	1.656*	1.170	1.283	0.112	0.952	1.153	1.301	0.148	0.996	
500		α	0.228	0.238	0.239	1.575*	1.455	1.593	0.138	0.969	1.433	1.615	0.182	0.995	
		θ	0.041	0.025	0.051	7.860*	3.554	4.243	0.689	0.953	3.445	4.352	0.907	0.991	
		β	1.161*	0.021	0.028	1.099*	1.181	1.270	0.088	0.946	1.167	1.284	0.116	0.987	
CL = 0.5		50	α	0.637	0.396	0.399	0.015	1.003	1.415	0.412	0.957	0.937	1.480	0.543	0.988
			θ	0.249	0.024	0.125	0.069	3.135	5.055	1.921	0.951	2.831	5.359	2.528	0.986
			β	6.503*	0.018	0.067	0.011	1.026	1.332	0.305	0.957	0.978	1.380	0.402	0.989
		100	α	0.635	0.397	0.398	7.529*	1.059	1.354	0.295	0.954	1.012	1.401	0.388	0.991
			θ	0.111	0.026	0.083	0.032	3.275	4.517	1.242	0.954	3.079	4.713	1.635	0.988
			β	3.710*	0.023	0.051	5.432*	1.066	1.279	0.213	0.955	1.032	1.313	0.280	0.986
		150	α	0.651	0.402	0.403	4.748*	1.081	1.309	0.228	0.959	1.045	1.345	0.300	0.992
			θ	0.071	0.017	0.067	0.021	3.426	4.437	1.011	0.951	3.266	4.597	1.331	0.987
			β	2.779*	0.022	0.044	3.709*	1.084	1.262	0.178	0.947	1.056	1.290	0.234	0.986
	300	α	0.647	0.402	0.402	2.494*	1.112	1.282	0.169	0.954	1.086	1.308	0.223	0.989	
		θ	0.047	0.031	0.054	0.010	3.526	4.224	0.698	0.954	3.416	4.335	0.919	0.992	
		β	1.772*	0.024	0.035	1.791*	1.111	1.232	0.122	0.939	1.092	1.252	0.160	0.992	
	500	α	0.651	0.403	0.403	1.463*	1.130	1.258	0.128	0.956	1.109	1.278	0.169	0.989	
		θ	0.035	0.032	0.047	6.113*	3.603	4.139	0.536	0.944	3.518	4.224	0.705	0.985	
		β	1.349*	0.023	0.031	1.067*	1.125	1.219	0.093	0.950	1.111	1.234	0.123	0.992	

* Indicate that the value multiply 10^{-3} . MSE = mean squared error; RB = relative bias; RE = relative error; SE = standard error; CP = coverage probability; CL = censoring level.

parameters are displayed in Table 4 and Table 5. The posterior kernel densities and dynamic trace for the unknown parameters (α, θ, β) are given in Figure 2 and Figure 3.

The approximate marginal posterior density of each parameter can be displayed in Figure 2. Trace

Table 2: Results of simulation study of MSE, RB, RE, SE, and confidence interval (CI) of different values of parameters case $I \equiv (\alpha = 1.5, \theta = 4, \beta = 1.2)$ for different samples size

n	ϕ	Properties				95%				99%					
		MSE	RB	RE	SE	Lower	Upper	Length	CP	Lower	Upper	Length	CP		
CL = 0.2	50	α	0.054	0.137	0.154	0.015	1.086	1.502	0.416	0.954	1.020	1.568	0.548	0.984	
		θ	0.710	0.100	0.211	0.105	2.943	5.853	2.910	0.960	2.483	6.314	3.831	0.986	
		β	0.027	0.113	0.136	0.013	1.157	1.514	0.357	0.951	1.100	1.571	0.470	0.985	
	100	α	0.054	0.147	0.155	7.390*	1.134	1.424	0.290	0.954	1.088	1.470	0.381	0.989	
		θ	0.345	0.072	0.147	0.051	3.286	5.292	2.005	0.957	2.969	5.609	2.640	0.986	
		β	0.023	0.113	0.125	6.476*	1.209	1.462	0.254	0.942	1.168	1.503	0.334	0.992	
	150	α	0.057	0.154	0.159	4.903*	1.151	1.386	0.235	0.949	1.114	1.424	0.310	0.987	
		θ	0.236	0.066	0.121	0.033	3.469	5.062	1.592	0.952	3.218	5.314	2.096	0.985	
		β	0.021	0.112	0.120	4.290*	1.232	1.438	0.206	0.955	1.199	1.470	0.271	0.990	
	300	α	0.057	0.156	0.159	2.383*	1.184	1.346	0.162	0.946	1.159	1.372	0.213	0.989	
		θ	0.123	0.054	0.088	0.016	3.672	4.758	1.086	0.954	3.500	4.929	1.429	0.990	
		β	0.019	0.109	0.114	2.288*	1.253	1.409	0.155	0.952	1.229	1.433	0.204	0.983	
	500	α	0.058	0.159	0.161	1.473*	1.197	1.326	0.129	0.952	1.176	1.346	0.170	0.990	
		θ	0.091	0.054	0.075	9.460*	3.801	4.630	0.829	0.947	3.670	4.761	1.092	0.993	
		β	0.019	0.111	0.114	1.287*	1.277	1.390	0.113	0.941	1.259	1.408	0.149	0.991	
	CL = 0.3	50	α	0.127	0.228	0.237	0.014	0.964	1.353	0.389	0.958	0.902	1.414	0.512	0.985
			θ	0.479	0.060	0.173	0.092	2.970	5.513	2.543	0.957	2.568	5.916	3.347	0.985
			β	0.016	0.066	0.106	0.014	1.085	1.474	0.388	0.959	1.024	1.535	0.511	0.991
		100	α	0.127	0.233	0.238	6.942*	1.014	1.286	0.272	0.948	0.971	1.329	0.358	0.992
			θ	0.184	0.013	0.107	0.043	3.215	4.887	1.671	0.955	0.951	5.151	2.200	0.985
			β	0.011	0.067	0.088	6.921*	1.145	1.416	0.271	0.944	1.102	1.459	0.357	0.987
		150	α	0.128	0.236	0.239	4.467*	1.039	1.253	0.214	0.948	1.005	1.287	0.282	0.989
			θ	0.120	0.018	0.086	0.028	3.412	4.736	1.324	0.953	0.202	4.945	1.743	0.986
			β	9.083*	0.065	0.079	4.511*	1.169	1.386	0.217	0.953	1.135	1.420	0.285	0.992
700		α	0.128	0.238	0.239	2.137*	1.071	1.216	0.145	0.948	1.048	1.239	0.191	0.985	
		θ	0.061	4.482*	0.062	0.014	3.537	4.499	0.962	0.950	3.385	4.651	1.266	0.991	
		β	8.074*	0.067	0.075	2.330*	1.201	1.359	0.158	0.947	1.176	1.384	0.208	0.994	
500		α	0.129	0.238	0.239	1.351*	1.084	1.202	0.118	0.952	1.065	1.221	0.156	0.988	
		θ	0.035	0.416*	0.047	8.347*	3.636	4.367	0.732	0.945	3.520	4.483	0.963	0.990	
		β	6.716*	0.063	0.068	1.363*	1.216	1.336	0.119	0.945	1.197	1.355	0.157	0.992	
CL = 0.5		50	α	0.357	0.394	0.398	0.011	0.750	1.066	0.316	0.948	0.700	1.116	0.416	0.993
			θ	0.286	0.043	0.134	0.072	3.180	5.165	1.985	0.946	2.866	5.479	2.613	0.982
			β	0.011	0.011	0.087	0.015	1.010	1.415	0.406	0.943	0.946	1.480	0.534	0.987
		100	α	0.355	0.395	0.397	5.685*	0.796	1.018	0.223	0.950	0.760	1.054	0.293	0.984
			θ	0.102	1.077*	0.080	0.032	3.369	4.623	1.254	0.958	3.171	4.821	1.650	0.993
			β	5.076*	1.354*	0.059	7.123*	1.062	1.341	0.279	0.955	1.018	1.385	0.368	0.994
		150	α	0.364	0.401	0.402	3.682*	0.810	0.987	0.177	0.952	0.782	1.015	0.233	0.989
			θ	0.072	4.204*	0.067	0.022	3.492	4.541	1.049	0.954	3.326	4.707	1.381	0.986
			β	3.390*	3.498*	0.049	4.742*	1.090	1.318	0.228	0.946	1.054	1.354	0.300	0.987
	300	α	0.365	0.402	0.403	1.839*	0.834	0.959	0.125	0.954	0.814	0.979	0.164	0.986	
		θ	0.036	0.015	0.048	0.010	3.585	4.297	0.711	0.959	3.473	4.409	0.936	0.989	
		β	1.700*	2.738*	0.034	2.373*	1.116	1.277	0.161	0.955	1.091	1.303	0.212	0.992	
	500	α	0.366	0.403	0.403	1.097*	0.847	0.944	0.096	0.952	0.832	0.959	0.127	0.991	
		θ	0.024	0.015	0.038	6.306*	3.662	4.215	0.553	0.954	3.575	4.302	0.728	0.992	
		β	1.161*	3.909*	0.028	1.510*	1.129	1.261	0.132	0.952	1.108	1.282	0.174	0.986	

* Indicate that the value multiply 10^{-3} . MSE = mean squared error; RB = relative bias; RE = relative error; SE = standard error; CP = coverage probability; CL = censoring level.

Plots of the Parameters sampled from posterior are shown in Figure 3.

The dynamic trace for parameters (α, θ, β) in Figure 3 show that the plots looks like a horizontal band, with no long upward or downward trends and the three chains are well-mixed and is indicative

Table 3: Results of simulation study of MSE, RB, RE, SE, and confidence interval (CI) of different values of parameters case $I \equiv (\alpha = 2, \theta = 4, \beta = 1.5)$ for different samples size

n	ϕ	Properties				95%				99%					
		MSE	RB	RE	SE	Lower	Upper	Length	CP	Lower	Upper	Length	CP		
CL = 0.2	50	α	0.196	0.220	0.221	5.999*	1.476	1.642	0.166	0.939	1.450	1.669	0.219	0.961	
		θ	0.506	0.128	0.178	0.070	2.519	4.460	1.942	0.955	2.211	4.767	2.556	0.983	
		β	0.017	0.073	0.086	9.468*	1.259	1.522	0.262	0.942	1.218	1.563	0.345	0.986	
	100	α	0.187	0.216	0.216	2.389*	1.522	1.615	0.094	0.937	1.507	1.630	0.123	0.956	
		θ	0.420	0.134	0.162	0.036	2.750	4.178	1.428	0.955	2.524	4.403	1.879	0.987	
		β	0.016	0.077	0.084	5.138*	1.283	1.485	0.201	0.961	1.252	1.517	0.265	0.988	
	150	α	0.184	0.214	0.215	1.478*	1.536	1.607	0.071	0.948	1.524	1.618	0.093	0.961	
		θ	0.415	0.142	0.161	0.025	2.841	4.021	1.180	0.948	2.654	4.208	1.554	0.990	
		β	0.016	0.080	0.084	3.442*	1.298	1.463	0.165	0.950	1.272	1.489	0.218	0.990	
	300	α	0.181	0.213	0.213	0.549*	1.556	1.593	0.037	0.956	1.550	1.599	0.049	0.966	
		θ	0.387	0.147	0.155	0.012	3.007	3.821	0.814	0.948	2.878	3.950	1.072	0.988	
		β	0.016	0.082	0.085	1.770*	1.316	1.437	0.120	0.955	1.297	1.456	0.158	0.991	
	500	α	0.180	0.212	0.212	0.297*	1.563	1.589	0.026	0.962	1.559	1.593	0.034	0.977	
		θ	0.385	0.150	0.155	7.351*	3.079	3.724	0.644	0.949	2.977	3.826	0.848	0.994	
		β	0.016	0.084	0.085	1.066*	1.328	1.421	0.093	0.954	1.313	1.436	0.123	0.993	
	CL = 0.3	50	α	0.284	0.262	0.266	0.013	1.293	1.658	0.366	0.955	1.235	1.716	0.481	0.990
			θ	0.473	0.118	0.172	0.071	2.546	4.513	1.967	0.956	2.235	4.825	2.590	0.985
			β	0.032	0.105	0.119	0.012	1.179	1.506	0.327	0.953	1.127	1.557	0.430	0.989
100		α	0.271	0.258	0.260	7.260*	1.342	1.627	0.285	0.970	1.297	1.672	0.375	0.994	
		θ	0.494	0.152	0.176	0.036	2.695	4.090	1.395	0.950	2.474	4.310	1.836	0.987	
		β	0.030	0.108	0.115	5.859*	1.223	1.453	0.230	0.953	1.187	1.489	0.302	0.994	
150		α	0.270	0.258	0.260	5.223*	1.359	1.610	0.251	0.973	1.319	1.649	0.330	0.997	
		θ	0.468	0.156	0.171	0.023	2.822	3.933	1.110	0.955	2.647	4.108	1.461	0.986	
		β	0.031	0.113	0.118	4.269*	1.228	1.433	0.205	0.952	1.196	1.466	0.270	0.987	
700		α	0.263	0.255	0.256	2.874*	1.392	1.587	0.195	0.984	1.361	1.618	0.257	0.996	
		θ	0.482	0.167	0.174	0.011	2.959	3.705	0.746	0.948	2.841	3.823	0.982	0.987	
		β	0.031	0.115	0.117	1.990*	1.261	1.396	0.135	0.944	1.239	1.417	0.178	0.992	
500		α	0.265	0.256	0.257	1.771*	1.410	1.565	0.155	0.940	1.385	1.589	0.204	0.998	
		θ	0.476	0.169	0.173	6.436*	3.043	3.607	0.564	0.954	2.954	3.696	0.743	0.993	
		β	0.030	0.114	0.116	1.191*	1.276	1.380	0.104	0.945	1.260	1.397	0.137	0.992	
CL = 0.5		50	α	0.696	0.414	0.417	0.015	0.962	1.383	0.421	0.954	0.896	1.450	0.554	0.989
			θ	0.248	0.069	0.125	0.059	2.911	4.533	1.622	0.956	2.654	4.790	2.135	0.985
			β	0.039	0.117	0.132	0.013	1.148	1.501	0.353	0.959	1.092	1.556	0.464	0.992
	100	α	0.693	0.415	0.416	7.417*	1.025	1.316	0.291	0.953	0.979	1.362	0.383	0.988	
		θ	0.278	0.112	0.132	0.028	3.007	4.097	1.091	0.951	2.834	4.270	1.436	0.992	
		β	0.039	0.125	0.132	6.667*	1.182	1.444	0.261	0.953	1.141	1.485	0.344	0.993	
	150	α	0.714	0.421	0.422	4.883*	1.040	1.275	0.234	0.949	1.003	1.312	0.309	0.986	
		θ	0.219	0.103	0.117	0.018	3.155	4.020	0.865	0.953	3.018	4.157	1.139	0.988	
		β	0.036	0.122	0.127	4.247*	1.215	1.419	0.204	0.954	1.183	1.452	0.268	0.989	
	300	α	0.709	0.421	0.421	2.335*	1.080	1.238	0.159	0.956	1.055	1.263	0.209	0.988	
		θ	0.243	0.117	0.123	9.146*	3.223	3.844	0.621	0.953	3.125	3.942	0.817	0.988	
		β	0.036	0.125	0.127	2.121*	1.241	1.385	0.144	0.942	1.218	1.407	0.190	0.990	
	500	α	0.713	0.422	0.422	1.455*	1.092	1.220	0.128	0.942	1.072	1.240	0.168	0.985	
		θ	0.238	0.118	0.122	5.497*	3.287	3.769	0.482	0.944	3.211	3.845	0.634	0.992	
		β	0.036	0.125	0.126	1.315*	1.255	1.371	0.115	0.948	1.237	1.389	0.152	0.988	

* Indicate that the value multiply 10^{-3} . MSE = mean squared error; RB = relative bias; RE = relative error; SE = standard error; CP = coverage probability; CL = censoring level.

of convergence.

The following conclusions can be observed on the properties of the estimated parameters from EW lifetime distribution under CS-PALT:

Table 4: The Bayes estimate of the unknown parameters case $I \equiv (\alpha = 0.3, \theta = 0.45, \beta = 1.6)$ for different sized sample with hyper parameter values ($\nu = 1.5, b = 1.25, c = 2, d = 1.75$).

	n	ϕ	Properties				95%			Start	Sample	
			Mean	SD	MC error	Median	Lower	Upper	Length			
CL = 0.4	10	α	0.6549	0.07380	0.001040	0.6547	0.5346	0.7757	0.2411	501	10500	
		θ	0.2172	0.02105	2.164E-4	0.2164	0.1841	0.2534	0.0693	501	10500	
		β	1.8290	0.15640	0.003156	1.7790	1.7050	2.1330	0.4280	501	10500	
	20	α	0.6142	0.02528	2.727E-4	0.6139	0.5729	0.6560	0.0831	501	10500	
		θ	0.4105	0.02191	2.668E-4	0.4103	0.3752	0.4464	0.0712	501	10500	
		β	1.7170	0.01729	2.645E-4	1.7120	1.7010	1.7510	0.0500	501	10500	
	30	α	0.5131	0.01738	2.239E-4	0.5130	0.4850	0.5421	0.0571	501	10500	
		θ	0.4014	0.01645	2.026E-4	0.4013	0.3753	0.4289	0.0536	501	10500	
		β	1.7060	0.00642	1.224E-4	1.7040	1.7000	1.7190	0.0190	501	10500	
	50	α	0.3084	0.00476	5.978E-5	0.3083	0.3007	0.3163	0.0156	501	10500	
		θ	0.3973	0.00847	1.096E-4	0.3973	0.3832	0.4110	0.0278	501	10500	
		β	1.7010	8.59E-4	1.504E-5	1.7010	1.7000	1.7030	0.0030	501	10500	
	100	α	0.2868	0.00225	2.398E-5	0.2867	0.2831	0.2905	0.0074	501	10500	
		θ	0.3973	0.00452	5.821E-5	0.3973	0.3899	0.4048	0.0149	501	10500	
		β	1.7000	1.84E-4	3.581E-6	1.7000	1.7000	1.7010	0.0010	501	10500	
	CL = 0.5	10	α	0.7041	0.07437	7.490E-4	0.7028	0.5815	0.8289	0.2474	501	10500
			θ	0.2176	0.02068	1.887E-4	0.2171	0.1848	0.2521	0.0673	501	10500
			β	1.7960	0.10540	0.001739	1.7630	1.7040	1.9960	0.2920	501	10500
20		α	0.6019	0.02399	2.669E-4	0.6018	0.5632	0.6416	0.0784	501	10500	
		θ	0.4566	0.02337	2.233E-4	0.4562	0.4186	0.4962	0.0776	501	10500	
		β	1.7200	0.02094	3.464E-4	1.7140	1.7010	1.7610	0.0600	501	10500	
30		α	0.5270	0.01692	1.684E-4	0.5267	0.4994	0.5550	0.0556	501	10500	
		θ	0.4542	0.01785	1.903E-4	0.4540	0.4252	0.4840	0.0588	501	10500	
		β	1.7070	0.00729	1.378E-4	1.7050	1.7000	1.7220	0.0220	501	10500	
50		α	0.2993	0.00483	5.848E-5	0.2992	0.2913	0.3072	0.0159	501	10500	
		θ	0.3736	0.00833	1.061E-4	0.3735	0.3599	0.3874	0.0275	501	10500	
		β	1.7010	9.11E-4	1.580E-5	1.7010	1.7000	1.7030	0.0030	501	10500	
100		α	0.2836	0.00227	2.639E-5	0.2836	0.2799	0.2874	0.0075	501	10500	
		θ	0.3997	0.00470	5.663E-5	0.3996	0.3920	0.4075	0.0155	501	10500	
		β	1.7000	1.99E-4	3.401E-6	1.7000	1.7000	1.7010	0.0010	501	10500	

CL = censoring level.

1. For all the cases, it is clear that MSEs, RB, RE, and SE decrease as sample size increases (Tables 1–3).
2. The MSEs of $\hat{\alpha}$ increase when the CL increases. But, the MSEs of $\hat{\theta}$ and $\hat{\beta}$ decrease when CL increases.
3. The RBs and REs of $\hat{\alpha}$ increase as the CL increases. Whereas, the RBs and REs of $\hat{\theta}$ and $\hat{\beta}$ decrease as the CL increases.
4. For fixed values of θ, β and as the value of α decreases, the MSEs, RBs, and REs of α decrease (Table 1 and Table 2).
5. For fixed values of θ, β and as the value of α decreases; the average length of confidence intervals (CIs) (at $\Upsilon = 0.95$ and 0.99) of $\hat{\alpha}$ decrease.
6. When sample size increase, the MSEs, RBs, REs for $\hat{\theta}$ and the length of confidence interval for unknown parameters (θ, β) increases (Table 1 and Table 2).

Table 5: The Bayes estimate of the unknown parameters case $I \equiv (\alpha = 0.4, \theta = 0.75, \beta = 1.6)$ for different sized sample with hyper parameter values ($\nu = 1.5, b = 1.25, c = 2, d = 1.75$)

	n	ϕ	Properties				95%			Start	Sample
			Mean	SD	MC error	Median	Lower	Upper	Length		
CL = 0.4	10	α	0.2021	0.02114	2.132E-4	0.2017	0.1682	0.2374	0.0692	501	10500
		θ	0.2488	0.02224	2.177E-4	0.2482	0.2135	0.2868	0.0733	501	10500
		β	2.6720	0.27920	0.004418	2.7460	2.0920	2.9820	0.8900	501	10500
	20	α	0.2228	0.00952	1.014E-4	0.2229	0.2069	0.2388	0.0319	501	10500
		θ	0.7208	0.03708	4.030E-4	0.7205	0.6605	0.7832	0.1227	501	10500
		β	1.7320	0.03266	6.086E-4	1.7220	1.7020	1.7960	0.0940	501	10500
	30	α	0.2810	0.00838	9.386E-5	0.2810	0.2673	0.2950	0.0277	501	10500
		θ	0.6216	0.02279	2.720E-4	0.6213	0.5847	0.6593	0.0746	501	10500
		β	1.7100	0.00992	1.618E-4	1.7070	1.7000	1.7300	0.0300	501	10500
	50	α	0.3599	0.00638	8.779E-5	0.3600	0.3495	0.3704	0.0209	501	10500
		θ	0.7683	0.01716	2.335E-4	0.7681	0.7404	0.7965	0.0561	501	10500
		β	1.7040	0.00354	5.780E-5	1.7020	1.7000	1.7100	0.0100	501	10500
	100	α	0.3283	0.00299	3.803E-5	0.3283	0.3234	0.3332	0.0098	501	10500
		θ	0.7716	0.00922	1.228E-4	0.7716	0.7566	0.7866	0.0300	501	10500
		β	1.7010	6.21E-4	1.065E-5	1.7000	1.7000	1.7020	0.0020	501	10500
CL = 0.5	10	α	0.2125	0.02164	2.168E-4	0.2121	0.1777	0.2492	0.0715	501	10500
		θ	0.2488	0.02191	2.285E-4	0.2480	0.2142	0.2861	0.0719	501	10500
		β	2.2740	0.37610	0.004562	2.2370	1.7430	2.9120	1.1690	501	10500
	20	α	0.2288	0.00962	1.126E-4	0.2288	0.2132	0.2448	0.0316	501	10500
		θ	0.8006	0.03868	4.545E-4	0.8004	0.7369	0.8650	0.1281	501	10500
		β	2.3690	0.35980	0.004547	2.3650	1.7910	2.9290	1.1380	501	10500
	30	α	0.2967	0.00809	9.804E-4	0.2967	0.2834	0.3103	0.0269	501	10500
		θ	0.7012	0.02429	2.923E-4	0.7007	0.6620	0.7423	0.0803	501	10500
		β	1.7130	0.01309	2.137E-4	1.7090	1.7010	1.7390	0.0380	501	10500
	50	α	0.3539	0.00658	8.562E-5	0.3538	0.3432	0.3648	0.0216	501	10500
		θ	0.7939	0.01819	2.421E-4	0.7939	0.7650	0.8240	0.0590	501	10500
		β	1.7070	0.00657	1.211E-4	1.7040	1.7000	1.7200	0.0200	501	10500
	100	α	0.3228	0.00291	3.645E-5	0.3228	0.3180	0.3276	0.0096	501	10500
		θ	0.7793	0.00917	9.365E-5	0.7792	0.7644	0.7944	0.0300	501	10500
		β	1.7010	8.25E-4	1.491E-5	1.7010	1.7000	1.7020	0.0020	501	10500

CL = censoring level.

- For fixed values of α, θ and as the value of β increases, the MSEs, RBs, REs, and the length of CIs (at $\Upsilon = 0.95$ and 0.99) for $(\hat{\alpha}, \hat{\theta}, \hat{\beta})$ increase as the sample size increases (Table 1 and Table 3).
- For fixed values of θ and as the value of α and β increases, the MSEs, RBs, REs, and the length of CIs (at $\Upsilon = 0.95$ and 0.99) of α and θ increase. But, the MSEs, RBs, REs, and the length of CIs at $\Upsilon = (0.95$ and $0.99)$ of θ and β decrease when sample size increase (Table 2 and Table 3).
- In case I, the maximum likelihood estimates have good statistical properties than the corresponding cases of parameters at $CL = 0.30$ and 0.50 (Tables 1–3).
- The length of the interval of all estimates decreases as the sample size increases. In addition; the length of the interval of the estimates at $\Upsilon = 0.95$ is smaller than the corresponding at $\Upsilon = 0.99$ (Tables 1–3).
- The Bayes estimate, SD and MCMC error decrease as sample size increases (Table 4 and Table 5).
- The Bayes estimates for case II have good statistical properties than case I at $CL = 0.50$ (Table 4 and Table 5).

Figure 2: Posterior kernel density for the parameters (α, θ, β) in different sample size and censored level (CL).

7. Conclusion

This paper presents the constant stress partially accelerated life test for EW distribution lifetime distribution and discuss the statistical inference based on multiple censored data. Both maximum likelihood and Bayes estimates are considered for the model parameters and accelerated factor. Additionally, the two-sided confidence limit and coverage probability of the model parameters are also constructed. The problem of Bayesian estimation of unknown parameters and accelerated factor under constant stress partially accelerated life test can be extended to include gamma and non-informative priors. In addition, other techniques such as MCMC technique are used to compute the approximate Bayes estimates and credible confidence intervals.

Figure 3: Posterior trace plots for the parameters (α, θ, β) in different sample size and censoring level (CL).

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